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SHORT PERIOD LUNAR AND SOLAR PERTURBATIONS FOR ARTIFICIAL SATELLITES

by

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SUMMARY

Two sets of expressions for the short period perturbations due to lunar and solar gravitational forces in the orbital elements of artificial satellites are derived. The first set of expressions applies to the case of small eccentricity while the second set is applicable to orbits of arbitrary eccentricity. The results are tabulated in a form suitable for programming on high speed computers.

CONTENTS

LIST OF SYMBOLS

A	$\frac{k^2 m'}{a'^{p'}}$ a^p $A'_{\nu q'}$ $A_{\nu q}$ $B_{i'q'}$ C_{iq}
$A_{\nu q}$	defined in tables 1-13
$A'_{\nu q'}$	defined in tables 1-13
a	semi-major axis of satellite orbit
a'	semi-major axis of the orbit of the disturbing body
$B_{i'q'}$	defined in tables 1-13
C	$\cos(\vec{r}, \vec{r}')$
C_{iq}	defined in tables 1-13
c	$\cos^2 I/2$
c'	$\cos^2 I'/2$
E	eccentric anomaly of the satellite
e	eccentricity of the satellite orbit
e'	eccentricity of the orbit of the disturbing body
f	true anomaly of the satellite
f	true anomaly of the disturbing body
G	$L(1 - e^2)^{1/2}$
g	argument of perigee of the satellite
H	$G \cos I$
h	right ascension of the node of the satellite
I	inclination of satellite orbit plane to earth's equator
I'	inclination of disturbing body's orbit plane to earth's equator
i	index on which disturbing function is summed
i'	index on which disturbing function is summed
k	Gaussian constant
L	$\mu^{1/2} a^{1/2}$
ℓ	mean anomaly of satellite
ℓ'	mean anomaly of disturbing body

m	n'/n
m'	mass of disturbing body
m_1	$m e$
$m_\oplus$	mass of the earth
n	mean motion of satellite
n'	mean motion of disturbing body
Q	determining function
q	index on which disturbing function is summed
q'	index on which disturbing function is summed
R	the disturbing function
R_P	the short-period disturbing function
R_S	$R - R_P$
$\bar{R}_S$	R_S with sine substituted for cosine
$\bar{r}$	(x,y,z), the radius vector to the satellite
$\bar{r}'$	(x',y',z'), the radius vector to the disturbing body
r	modulus of r
r'	modulus of r'
α	$q\,f + q\omega + q'\omega' + \nu\theta$
β	$e/(1 + \sqrt{1 - e^2})$
γ	$\sin^2 I/2$
γ'	$\sin^2 I'/2$
θ	$\Omega - \Omega'$
μ	$k^2\, m'$
ν	index on which disturbing function is summed
τ	$i'\,\ell' + q\omega + q'\omega' + \nu\theta$
ϕ	$q\omega + q'\omega' + \nu\theta$

Ω right ascension of the node of the satellite

Ω' right ascension of the node of the disturbing body

ω argument of perigee of the satellite

ω' argument of perigee of the disturbing body

SHORT PERIOD LUNAR AND SOLAR PERTURBATIONS FOR ARTIFICIAL SATELLITES

INTRODUCTION

The influence of lunar and solar gravitational forces on the motion of a satellite were recognized shortly after the launchings of the first satellites (References 1 and 2). This led to analytic formulations for the perturbations in the elements. By expanding the disturbing function in powers of eccentricity and in terms of the mean anomalies of the satellite and perturbing body, accurate representations for the long period motion were obtained in Reference 1. In addition, the results of Reference 2 are useful for satellites of small eccentricities.

In order to accurately describe the short-period perturbations of satellites in orbits of moderate to high as well as low eccentricity, the results given in References 1 and 2 are both modified and extended in this report.

In obtaining the solution for the case of low to moderate eccentricity, the expansion of the disturbing function in terms of the mean anomaly of the satellite as adapted in Reference 2 is carried out. The solution for satellites of arbitrary eccentricity is derived separately. Following the suggestion in Reference 2, the mean anomaly of the satellite is replaced by its eccentric anomaly. This permits replacing the infinite series in terms of the eccentricity of the satellite which may converge slowly, by polynomial in closed form. In addition, the convenient tabular form for representing the expansions is also adapted.

However, rather than tabulating the disturbing function as done in References 1 and 2, in this report the determining function (see Reference 3), and its differential coefficients are given. In this way, the need for setting up the variation equations and performing the integrations by the reader is avoided. The perturbations in the elements are found by forming linear combinations of the tabulated coefficients.

THE DISTURBING FUNCTION

From Reference 4 the disturbing function, R , for the gravitational forces acting on a satellite due to the presence of a third body is given by

$$R = \frac{k^2 m'}{a'^3} a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \left(\frac{3}{2} C^2 - \frac{1}{2}\right) + \frac{Gm'}{a'^4} a^3 \left(\frac{r}{a}\right)^3 \left(\frac{a'}{r'}\right)^4 \left(\frac{5}{2} C^3 - \frac{3}{2} C\right) \tag{1}$$

where $\quad k$ = Gaussian constant $[k^2 = 6.670 \times 10^{-5}\ (\mathrm{cm}^3/\mathrm{kg.\ sec}^2)]$

m' = the mass of the disturbing body

$m' = 7.35 \times 20^{22}$ kg.
$m' = 1.99 \times 10^{30}$ kg.

a, a' = the semi-major axis of the satellite and the disturbing body respectively.

$a' = 60.266011$ earth radii for the moon
$a' = 23438.524$ earth radii for the sun

r, r' = absolute value of radius vector from the center of the earth to the satellite and to the disturbing body respectively

C = cosine of the angle between $\vec{r}$ and $\vec{r}'$;

C is given by

$$C = \frac{xx' + yy' + zz'}{rr'} . \tag{2}$$

By substituting the expression for C given by Equation (2) into the disturbing function, R can be written in the form

$$R = \frac{k^2 m'}{a'^3} \left\{ a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \left[\frac{1}{4} \frac{r^2 - 3z^2}{r^2} \frac{r'^2 - 3z'^2}{r'^2} \right.\right.$$

$$\left.\left. + \frac{3}{4} \frac{x^2 - y^2}{r^2} \frac{x'^2 - y'^2}{r'^2} + 3 \frac{xy}{r^2} \frac{x'y'}{r'^2} + 3 \frac{yz}{r^2} \frac{y'z'}{r'^2} \right] \right\}$$

$$+ \frac{k^2 m'}{a'^4} \left\{ a^3 \left(\frac{r}{a}\right)^3 \left(\frac{a'}{r'}\right)^4 \left[\frac{3}{8} \frac{r^2 x - 5xz^2}{r^3} \frac{r'^2 x' - 5x'z'^2}{r'^3} \right.\right.$$

$$+ \frac{3}{8} \frac{r^2 y - 5yz^2}{r^3} \frac{r'^2 y' - 5y'z'^2}{r'^3}$$

$$+\frac{1}{4}\,\frac{3r^2z-5z^3}{r^3}\,\frac{3r'^2z'-5z'^3}{r'^3}+\frac{5}{8}\,\frac{x^3-3xy^2}{r^3}\,\frac{x'^3-3x'y'^2}{r'^3}$$

$$\left.+\frac{5}{8}\,\frac{3x^2y-y^3}{r^3}\,\frac{3x'^2y'-y'^3}{r'^3}+\frac{15}{4}\,\frac{x^2z-y^2z}{r^3}\,\frac{x'^2z'-y'^2z'}{r'^3}+15\,\frac{xyz}{r^3}\,\frac{x'y'z'}{r'^3}\right]\Bigg\} \quad (3)$$

Next, the quantities enclosed in the brackets are expressed in terms of the orbital elements of the disturbed and disturbing bodies by the formulas

$$\begin{aligned}
\frac{x}{r} &= c^2\cos(f+\omega+\theta)+\gamma^2\cos(f+\omega-\theta)\\
\frac{y}{r} &= c^2\sin(f+\omega+\theta)-\gamma^2\sin(f+\omega-\theta)\\
\frac{z}{r} &= 2\gamma c\sin(f+\omega)\\
\frac{x'}{r'} &= \cos(f'+\omega')\\
\frac{y'}{r'} &= (1-2\gamma'^2)\sin(f'+\omega')\\
\frac{z'}{r'} &= 2\gamma'c'\sin(f'+\omega')
\end{aligned} \quad (4)$$

where c = $\cos^2 I/2$ where I equals the angle of inclination of the orbit plane of the satellite to equatorial plane of the earth

θ = $\Omega - \Omega'$ where Ω and Ω' are the longitude of the ascending node of the satellite and of the disturbing body respectively

γ = $\sin^2 I/2$

f = true anomaly of the satellite

ω = argument of perigee of the satellite.

and where the primed elements refer to the corresponding elements of the disturbing body.

The disturbing function may then be written in the form

$$R = \frac{k^2 m'}{a'^3} a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \sum A'_{\nu q'} A_{\nu q} \cos(qf + q'f' + q\omega + q'\omega' + \nu\theta)$$
$$+ \frac{k^2 m'}{a'^4} a^3 \left(\frac{r}{a}\right)^3 \left(\frac{a}{r'}\right)^4 \sum A'_{\nu q'} A_{\nu q} \cos(qf + q'f' + q\omega + q'\omega' + \nu\theta) \tag{5}$$

In Equation (5) $A_{\nu q}$ is a function of c and γ only while $A'_{\nu q'}$ is a function of c' and γ'. In the first summation of Equation (5) q and q' take on the five sets of values (0,0), (2,-2), (2,0), (0,2), and (2,2). In the second summation q and q' take on the eight values (1,1), (1,-1), (1,3), (1,-3), (3,1), (3,-1), (3,3) and (3,-3).

Next the development proceeds in terms of the angular variables that are dependent upon the mean anomaly of the satellite or of the disturbing body. In Equation (5) first hold q, q', and ν constant and let

$$\alpha = qf + q\omega + q'\omega' + \nu\theta \tag{6}$$

to obtain a term of the form

$$\frac{k^2 m'}{a'^{p'}} a^{p'} \left(\frac{r}{a}\right)^p A'_{\nu q} A_{\nu q} \left[\left(\frac{a'}{r'}\right)^{p'} \cos(\alpha + q'f')\right] \tag{7}$$

The part enclosed by the brackets may be expanded through the use of Cayley's Tables (Reference 5) or by formulas from elliptic motion in Reference 4 into a series of the form

$$\frac{k^2 m'}{a'^{p'}} a^p \left(\frac{r}{a}\right)^p A'_{\nu q'} A_{\nu q} \sum_{i'} B_{i'q'} \cos(\alpha + i'\ell') \tag{8}$$

In (8) i' takes on both positive and negative integers and zero values. Also $B_{i'q'}$ is a power series in the eccentricity e' of the disturbing body, which has a value of approximately .055 for the moon and .017 for the sun.

The procedure outlined above may be continued for particular values of i'. Each term of the series, (8), may then be written in the form

$$\frac{k^2 m'}{a'^{p'}} a^p A'_{\nu q'} A_{\nu q} B_{i'q'} \left[\left(\frac{r}{a}\right)^p \cos (q f + \tau)\right] \tag{9}$$

where $\tau = i'\ell' + q\omega + q'\omega + \nu\theta$

For satellites in orbits of small eccentricity an expansion of the quantity in brackets in (9) converges rapidly. However, when the eccentricity of the satellite's orbit is large it may be more convenient to follow the suggestion of Reference 2 and expand this quantity in terms of the eccentric anomaly of the satellite. Therefore, two distinct approaches to the solution will be pursued. The first case will be that of small eccentricity with the quantity in brackets in (9) expressed in terms of a truncated infinite series in eccentricity and mean anomaly, while in the second case that of arbitrary eccentricity this quantity is expressed in terms of a finite series of eccentric anomaly with exact coefficients involving the eccentricity.

While it is true that the solution for the case of arbitrary eccentricity is valid for small eccentricity also, the solution for the small eccentricity case is much simpler. Thus, if the satellite for which the short period lunar and solar perturbations are to be computed is one of small eccentricity considerable computing time and labor may be avoided by using the formulas for that case rather than those for the case or arbitrary eccentricity.

THE DISTURBING FUNCTION FOR LOW ECCENTRICITY

By means of formulas in Reference 5, (9) may be expressed in the following form

$$\frac{k^2 m'}{a'^{p'}} a^p A'_{\nu q'} A_{\nu q} B_{i'q'} \sum_i C_{iq} \cos (i\ell + i'\ell' + q\omega + q'\omega' + \nu\theta) \tag{10}$$

where C_{iq} is a function of the eccentricity of the satellite. The coefficients $A_{\nu q}$, $A'_{\nu q'}$, $B_{i'q'}$, and C_{iq} are given in Tables 1 through 13 for low eccentricity. Thus, for this case the mean anomalies of both the satellite and the disturbing body are used in the expansion of the disturbing function and later the determining function. However, if $i = i' = 0$, then that particular term is not a short period term and should be omitted. Since it would be inconvenient to list the terms for which $i = 0$ and $i' \neq 0$ and $i \neq 0$, $i' = 0$ separately from the tables, an asterisk (*) is placed next to the entry $i = 0$ or $i' = 0$ to remind the reader to omit the term for which both i and i' are zero.

THE DETERMINING FUNCTION AND PERTURBATIONS FOR LOW ECCENTRICITY

Let Q denote the first order part of the determining function (Reference 3), then the perturbations in the Delaunay Elements L, G, H, ℓ, g, h are given by

$$\delta L = \frac{\partial Q}{\partial \ell} \qquad \delta \ell = -\frac{\partial Q}{\partial L}$$

$$\delta G = \frac{\partial Q}{\partial g} \qquad \delta g = -\frac{\partial Q}{\partial G} \tag{11}$$

$$\delta H = \frac{\partial Q}{\partial h} \qquad \delta h = -\frac{\partial Q}{\partial H}$$

where

$L = \mu^{1/2} a^{1/2}$ — ℓ = mean anomaly

$G = L(1 - e^2)^{1/2}$ — g = argument of perigee

$H = G \cos I$ — h = longitude of ascending node

In this report g is replaced by ω, while h and Ω are equivalent. Also

$$\cos I = 1 - 2\gamma^2$$

From Reference 3, we find that Q must satisfy the partial differential equation

$$n \frac{\partial Q}{\partial \ell} + n' \frac{\partial Q}{\partial \ell'} = R_p \tag{12}$$

where

n = mean motion of the mean anomaly of the satellite

n' = mean motion of the mean anomaly of the disturbing body

n' = 13.2/day for the moon (approximately)

n' = .986/day for the sun (approximately)

R_p = Periodic part of the disturbing function.

After integration, Equation (12) consists of two groups which are of the form,

$$Q = \frac{k^2}{a'^{p'}} a^p \sum_{\nu} \sum_{i} \sum_{i'} A'_{\nu q'} A_{\nu q} B_{i'q'} C_{iq} \frac{\sin(i\ell + i'\ell' + \nu\theta + q\omega + q'\omega')}{in + i'n'} \quad (13)$$

In group 1 $(p, p') = (2,3)$ and in group 2, $(p, p') = (3,4)$.

A typical term of R_p is of the form

$$A \cos(i\ell + i'\ell' + \phi)$$

and the perturbations then resulting from such a term are obtained from

$$\delta L = \frac{\partial Q}{\partial \ell}$$
$$\delta G = \frac{\partial Q}{\partial g} \quad (14)$$
$$\delta H = \frac{\partial Q}{\partial \theta}$$

$$\delta \ell = -\frac{\partial Q}{\partial L} = -\frac{2L}{\mu}\frac{\partial Q}{\partial a} + \frac{G^2}{eL^3}\frac{\partial Q}{\partial e}$$
$$\delta g = -\frac{\partial Q}{\partial G} = \frac{G}{eL^2}\frac{\partial Q}{\partial e} + \frac{\cos I}{G}\frac{\partial Q}{\partial(\cos I)} = \frac{G}{eL^2}\frac{\partial Q}{\partial e} - \frac{(1-2\gamma^2)}{4G\gamma}\frac{\partial Q}{\partial \gamma} \quad (15)$$
$$\delta h = -\frac{\partial Q}{\partial H} = \frac{1}{G}\frac{\partial Q}{\partial(\cos I)} = \frac{1}{4G\gamma}\frac{\partial Q}{\partial \gamma}$$

In order to obtain $(\partial Q/\partial a)$ one must make use of

$$\frac{dn}{da} = -\frac{3}{2}\frac{n}{a}. \tag{16}$$

Therefore, if the typical term in R_p is $A\cos(i\ell + i'\ell' + \phi)$ where

$$A = \frac{k^2 m}{a'^{p'}} a^p A'_{\nu q'} A_{\nu q} B_{i'q'} C_{iq},$$

then the perturbation may be obtained by applying the following formulas

$$\begin{aligned}
\frac{\partial B}{\partial a} &= \frac{k^2 m}{a'^{p'}} A'_{\nu q'} A_{\nu q} B_{i'q'} C_{iq} \frac{a^{p-1}}{(in + i'n')^2} [(2p+3)\, in + 2pi'n'] \\
\frac{\partial B}{\partial e} &= \frac{k^2 m}{a'^{p'}} A'_{\nu q'} B_{i'q'} \frac{\partial C_{iq}}{\partial e} \frac{1}{in + i'n'} \\
\frac{\partial B}{\partial \gamma} &= \frac{k^2 m}{a'^{p'}} A'_{\nu q'} \frac{\partial A_{\nu q}}{\partial \gamma} B_{i'q'} C_{iq} \frac{1}{in + i'n'}
\end{aligned} \tag{17}$$

where

$$B = \frac{A}{in + i'n'}$$

ILLUSTRATIVE EXAMPLE OF THE USE OF THE TABLES FOR LOW ECCENTRICITY

In order to illustrate the use of the tables, the contribution to the perturbation in ℓ resulting from the term defined by the indices $q = 3$, $q' = 3$, $\nu = 2$, $i' = 5$, and $i = 2$ is obtained. From the first of Equations (15)

$$\delta\ell = -\frac{2L}{\mu}\frac{\partial Q}{\partial a} + \frac{G^2}{eL^3}\frac{\partial Q}{\partial e},$$

and from the first two of Equations (17), where A represents a single coefficient in the series for Q

$$\frac{\partial A}{\partial a} = \frac{k^2 m}{a^4} \; A'_{23}\, A_{23}\, B_{53}\, C_{23} \; \frac{2a^2}{(2n + 5n')^2} \; [7n + 15\, n']$$

$$\frac{\partial A}{\partial e} = \frac{k^2 m}{a^4} \; A'_{23}\, A_{23}\, B_{53} \; \frac{\partial C_{23}}{\partial e} \; \frac{1}{2n + 5n'} \; .$$

From Table 12

$$A'_{23} = -\frac{15}{4}\, c'\, \gamma'^5$$

$$A_{23} = c^5\, \gamma$$

$$B_{53} = \frac{127}{8}\, e'^2$$

$$C_{23} = -\frac{9}{2}\, e + \frac{33}{4}\, e^3$$

$$\frac{\partial C_{23}}{\partial e} = -\frac{9}{2} + \frac{99}{4}\, e^2$$

Therefore, due to this single term

$$\delta \ell = \left(-\frac{2L}{\mu}\, \frac{\partial A}{\partial a} + \frac{G^2}{eL^3}\, \frac{\partial A}{\partial e} \right) \; \cos \; (2\ell + 5\ell' + 3\omega + 3\omega' + 2\theta) \, .$$

LOW ECCENTRICITY

TABLE 1

$q = 0, \quad q' = 0$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{1}{4}(1 - 6\gamma'^2 + 6\gamma'^4)$	$1 - 6\gamma^2 + 6\gamma^4$	$-12\gamma(1-2\gamma^2)$	0
$3c'\gamma'(1 - 2\gamma'^2)$	$c\gamma(1-2\gamma^2)$	$c^{-1}(1 - 8\gamma^2 + 8\gamma^4)$	1
$3c'^2\gamma'^2$	$c^2\gamma^2$	$2\gamma(1-2\gamma^2)$	2
	$B_{i'q'}$		i'
	$\frac{9}{4}e'^2 + \frac{7}{4}e'^4$		-2
	$\frac{3}{2}e' + \frac{27}{16}e'^3$		-1
	$(1 - e'^2)^{-3/2}$		0*
	$\frac{3}{2}e' + \frac{27}{16}e'^3$		1
	$\frac{9}{4}e'^2 + \frac{7}{4}e'^4$		2
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$-\frac{1}{12}e^4$	$-\frac{e^3}{3}$	-4
	$-\frac{1}{8}e^3$	$-\frac{3}{8}e^2$	-3
	$-\frac{1}{4}e^2 + \frac{1}{12}e^4$	$-\frac{e}{2} + \frac{e^3}{3}$	-2
	$-e + \frac{1}{8}e^3$	$-1 + \frac{3}{8}e^2$	-1
	$1 + \frac{3}{2}e^2$	$3e$	0*
	$-e + \frac{1}{8}e^3$	$-1 + \frac{3}{8}e^2$	1
	$-\frac{1}{4}e^2 + \frac{1}{12}e^4$	$-\frac{e}{2} + \frac{e^3}{3}$	2
	$-\frac{1}{8}e^3$	$-\frac{3}{8}e^2$	3
	$-\frac{1}{12}e^4$	$-\frac{e^3}{3}$	4

TABLE 2

$q = 2, \quad q' = -2$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{3}{4}\gamma'^4$	γ^4	$4\gamma^3$	-2
$3c'\gamma'^3$	$c\gamma^3$	$c^{-1}\gamma^2(3-4\gamma^2)$	-1
$\frac{9}{2}c'^2\gamma'^2$	$c^2\gamma^2$	$2\gamma(1-2\gamma^2)$	0
$3c'^3\gamma'$	$c^3\gamma$	$c(1-4\gamma^2)$	1
$\frac{3}{4}c'^4$	c^4	$-4c^2\gamma$	2
	$B_{i'q'}$		i'
	$\frac{17}{2}e'^2 - \frac{115}{6}e'^4$		-4
	$\frac{7}{2}e' - \frac{123}{16}e'^3$		-3
	$1 - \frac{5}{2}e'^2 + \frac{13}{16}e'^4$		-2
	$-\frac{1}{2}e' + \frac{1}{16}e'^3$		-1
	0		0*
	$\frac{1}{48}e'^3$		1
	$\frac{1}{24}e'^4$		2
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$-\frac{1}{16}e^4$	$-\frac{1}{4}e^3$	-2
	$-\frac{7}{24}e^3$	$-\frac{7}{8}e^2$	-1
	$\frac{5}{2}e^2$	$5e$	0*
	$-3e + \frac{13}{8}e^3$	$-3 + \frac{39}{8}e^2$	1
	$1 - \frac{5}{2}e^2 + \frac{23}{16}e^4$	$-5e + \frac{23}{4}e^3$	2
	$e - \frac{19}{8}e^3$	$1 - \frac{57}{8}e^2$	3
	$e^2 - \frac{5}{2}e^4$	$2e - 10e^3$	4
	$\frac{25}{24}e^3$	$\frac{25}{8}e^2$	5
	$\frac{9}{8}e^4$	$\frac{9}{2}e^3$	6

LOW ECCENTRICITY

TABLE 3

$q = 2, \quad q' = 0$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{3}{2} c'^2 \gamma'^2$	γ^4	$4 \gamma^3$	-2
$3 c' \gamma' (1 - 2 \gamma'^2)$	$c \gamma^3$	$c^{-1} \gamma^2 (3 - 4 \gamma^2)$	-1
$\frac{3}{2} (1 - 6 \gamma'^2 + 6 \gamma'^4)$	$c^2 \gamma^2$	$2 \gamma (1 - 2 \gamma^2)$	0
$-3 c' \gamma' (1 - 2 \gamma'^2)$	$c^3 \gamma$	$c (1 - 4 \gamma^2)$	1
$\frac{3}{2} c'^2 \gamma'^2$	c^4	$-4 c^2 \gamma$	2
	$B_{i'q'}$		i'
	$\frac{9}{4} e'^2 + \frac{7}{4} e'^4$		-2
	$\frac{3}{2} e' + \frac{27}{16} e'^3$		-1
	$1 + \frac{3}{2} e'^2 + \frac{15}{8} e'^4$		0*
	$\frac{3}{2} e' + \frac{27}{16} e'^3$		1
	$\frac{9}{4} e'^2 + \frac{7}{4} e'^4$		2
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$-\frac{1}{16} e^4$	$-\frac{1}{4} e^3$	-2
	$-\frac{7}{24} e^3$	$-\frac{7}{8} e^2$	-1
	$\frac{5}{2} e^2$	$5 e$	0*
	$-3 e + \frac{13}{8} e^3$	$-3 + \frac{39}{8} e^2$	1
	$1 - \frac{5}{2} e^2 + \frac{23}{16} e^4$	$-5 e + \frac{23}{4} e^3$	2
	$e - \frac{19}{8} e^3$	$1 - \frac{57}{8} e^2$	3
	$e^2 - \frac{5}{2} e^4$	$2 e - 10 e^3$	4
	$\frac{25}{24} e^3$	$\frac{25}{8} e^2$	5
	$\frac{9}{8} e^4$	$\frac{9}{2} e^3$	6

LOW ECCENTRICITY

TABLE 4

$q = 0, \; q' = 2$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{3}{2} c'^4$	$c^2 \gamma^2$	$2\gamma(1 - 2\gamma^2)$	-2
$-3c'^3 \gamma'$	$c\gamma(1 - 2\gamma^2)$	$c^{-1}(1 - 8\gamma^2 + 8\gamma^4)$	-1
$\frac{3}{2} c'^2 \gamma'^2$	$1 - 6\gamma^2 + 6\gamma^4$	$-12\gamma(1 - 2\gamma^2)$	0
$3c' \gamma'^3$	$c\gamma(1 - 2\gamma^2)$	$c^{-1}(1 - 8\gamma^2 + 8\gamma^4)$	1
$\frac{3}{2} \gamma'^4$	$c^2 \gamma^2$	$2\gamma(1 - 2\gamma^2)$	2
	$B_{i'q'}$		i'
	$\frac{1}{24} e'^4$		-2
	$\frac{1}{48} e'^3$		-1
	0		0*
	$-\frac{1}{2} e' + \frac{1}{16} e'^3$		1
	$1 - \frac{5}{2} e'^2 + \frac{13}{16} e'^4$		2
	$\frac{7}{2} e' - \frac{123}{16} e'^3$		3
	$\frac{17}{2} e'^2 - \frac{115}{16} e'^4$		4
	C_{iq}	$\frac{\partial C_{iq}}{de}$	i
	$-\frac{1}{12} e^4$	$-\frac{e^3}{3}$	-4
	$-\frac{1}{8} e^3$	$-\frac{3}{8} e^2$	-3
	$-\frac{1}{4} e^2 + \frac{1}{12} e^4$	$-\frac{e}{2} + \frac{e^3}{3}$	-2
	$-e + \frac{1}{8} e^3$	$-1 + \frac{3}{8} e^2$	-1
	$1 + \frac{3}{2} e^2$	$3e$	0*
	$-e + \frac{1}{8} e^3$	$-1 + \frac{3}{8} e^2$	1
	$-\frac{1}{4} e^2 + \frac{1}{12} e^4$	$-\frac{e}{2} + \frac{e^3}{3}$	2
	$-\frac{1}{8} e^3$	$-\frac{3}{8} e^2$	3
	$-\frac{1}{12} e^4$	$-\frac{e^3}{3}$	4

TABLE 5
$q = 2, \; q' = 2$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{3}{4} c'^4$	γ^4	$4 \gamma^3$	-2
$-3 c'^3 \gamma'$	$c \gamma^3$	$c^{-1} \gamma^2 (3 - 4 \gamma^2)$	-1
$\frac{9}{2} c'^2 \gamma'^2$	$c^2 \gamma^2$	$2 \gamma (1 - 2 \gamma^2)$	0
$-3 c' \gamma'^3$	$c^3 \gamma$	$c (1 - 4 \gamma^2)$	1
$\frac{3}{4} \gamma'^4$	c^4	$-4 c^2 \gamma$	2
	$B_{i'q'}$		i'
	$\frac{1}{24} e'^4$		-2
	$\frac{1}{48} e'^3$		-1
	0		0^*
	$-\frac{1}{2} e' + \frac{1}{16} e'^3$		1
	$1 - \frac{5}{2} e'^2 + \frac{13}{16} e'^4$		2
	$\frac{7}{2} e' - \frac{123}{16} e'^3$		3
	$\frac{17}{2} e'^2 - \frac{115}{16} e'^4$		4
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$-\frac{1}{16} e^4$	$-\frac{1}{4} e^3$	-2
	$-\frac{7}{24} e^3$	$-\frac{7}{8} e^2$	-1
	$\frac{5}{2} e^2$	$5 e$	0^*
	$-3 e + \frac{13}{8} e^3$	$-3 + \frac{39}{8} e^2$	1
	$1 - \frac{5}{2} e^2 + \frac{23}{16} e^4$	$-5 e + \frac{23}{4} e^3$	2
	$e - \frac{19}{8} e^3$	$1 - \frac{57}{8} e^2$	3
	$e^2 - \frac{5}{2} e^4$	$2 e - 10 e^3$	4
	$\frac{25}{24} e^3$	$\frac{25}{8} e^2$	5
	$\frac{9}{8} e^4$	$\frac{9}{2} e^3$	6

LOW ECCENTRICITY

TABLE 6

$q = 1, \; q' = 1$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{45}{8} c'^4 \gamma'^2$	$c^2 \gamma^4$	$2\gamma^3 (2 - 3\gamma^2)$	-3
$\frac{15}{4} c'^3 (1 - 3\gamma'^2)\gamma'$	$c\gamma^3 (2 - 3\gamma^2)$	$c^{-1}\gamma^2 (6 - 23\gamma^2 + 18\gamma^4)$	-2
$\frac{3}{8} c'^2 (1 - 10\gamma'^2 + 15\gamma'^4)$	$\gamma^2 (6 - 20\gamma^2 + 15\gamma^4)$	$2\gamma (6 - 40\gamma^2 + 45\gamma^4)$	-1
$-\frac{9}{2} c'\gamma' (1 - 5\gamma'^2 + 5\gamma'^4)$	$c\gamma (1 - 5\gamma^2 + 5\gamma^4)$	$c^{-1} (1 - 17\gamma^2 + 45\gamma^4 - 30\gamma^6)$	0
$\frac{3}{8}\gamma'^2 (6 - 20\gamma'^2 + 15\gamma'^4)$	$c^2 (1 - 10\gamma^2 + 15\gamma^4)$	$-2\gamma (11 - 50\gamma^2 + 45\gamma^4)$	1
$\frac{15}{4} c'\gamma'^3 (2 - 3\gamma'^2)$	$c^3 (1 - 3\gamma^2)\gamma$	$c (1 - 13\gamma^2 + 18\gamma^4)$	2
$\frac{45}{8} c'^2 \gamma'^4$	$c^4 \gamma^2$	$2c^2\gamma (1 - 3\gamma^2)$	3
	$B_{i'q'}$		i'
	$\frac{11}{8} e'^2$		-1
	e'		0*
	$1 + 2e'^2$		1
	$3e'$		2
	$\frac{53}{8} e'^2$		3
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$\frac{7}{128} e^4$	$\frac{7}{32} e^3$	-3
	$\frac{1}{6} e^3$	$\frac{1}{2} e^2$	-2
	$\frac{11}{8} e^2 + \frac{7}{48} e^4$	$\frac{11}{4} e + \frac{7}{12} e^3$	-1
	$-\frac{5}{2} e - \frac{15}{8} e^3$	$-\frac{5}{2} - \frac{45}{8} e^2$	0*
	$1 + 2e^2 - \frac{41}{64} e^4$	$4e - \frac{41}{16} e^3$	1
	$-\frac{e}{2} + e^3$	$-\frac{1}{2} + 3e^2$	2
	$-\frac{3}{8} e^2 + \frac{11}{16} e^4$	$-\frac{3}{4} e + \frac{11}{4} e^3$	3
	$-\frac{7}{24} e^3$	$-\frac{7}{8} e^2$	4
	$-\frac{95}{384} e^4$	$-\frac{95}{96} e^3$	5

LOW ECCENTRICITY

TABLE 7

$q = 1,\ q' = -1$

$A'_{\nu q'}$	$A_{\nu q}$	$\dfrac{dA_{\nu q}}{d\gamma}$	ν
$\frac{45}{8} c'^2 \gamma'^4$	$c^2 \gamma^4$	$2 \gamma^3 (2 - 3 \gamma^2)$	-3
$\frac{15}{4} c' \gamma'^3 (2 - 3 \gamma'^2)$	$c \gamma^3 (2 - 3 \gamma^2)$	$c^{-1} \gamma^2 (6 - 23 \gamma^2 + 18 \gamma^4)$	-2
$\frac{3}{8} \gamma'^2 (6 - 20 \gamma'^2 + 15 \gamma'^4)$	$\gamma^2 (6 - 20 \gamma^2 + 15 \gamma^4)$	$2 \gamma (6 - 40 \gamma^2 + 45 \gamma^4)$	-1
$\frac{9}{2} c' \gamma' (1 - 5 \gamma'^2 + 5 \gamma'^4)$	$c \gamma (1 - 5 \gamma^2 + 5 \gamma^4)$	$c^{-1} (1 - 17 \gamma^2 + 45 \gamma^4 - 30 \gamma^6)$	0
$\frac{3}{8} c'^2 (1 - 10 \gamma'^2 + 15 \gamma'^4)$	$c^2 (1 - 10 \gamma^2 + 15 \gamma^4)$	$-2 \gamma (11 - 50 \gamma^2 + 45 \gamma^4)$	1
$\frac{15}{4} c'^3 (1 - 3 \gamma'^2) \gamma'$	$c^3 (1 - 3 \gamma^2) \gamma$	$c (1 - 13 \gamma^2 + 18 \gamma^4)$	2
$\frac{45}{8} c'^4 \gamma'^2$	$c^4 \gamma^2$	$2 c^2 \gamma (1 - 3 \gamma^2)$	3
	$B_{i'q'}$		i'
	$\frac{53}{8} e'^2$		-3
	$3 e'$		-2
	$1 + 2 e'^2$		-1
	e'		0*
	$\frac{11}{8} e'^2$		1
	C_{iq}	$\dfrac{dC_{iq}}{de}$	i
	$\frac{7}{128} e^4$	$\frac{7}{32} e^3$	-3
	$\frac{1}{6} e^3$	$\frac{1}{2} e^2$	-2
	$\frac{11}{8} e^2 + \frac{7}{48} e^4$	$\frac{11}{4} e + \frac{7}{12} e^3$	-1
	$-\frac{5}{2} e - \frac{15}{8} e^3$	$-\frac{5}{2} - \frac{45}{8} e^2$	0*
	$1 + 2 e^2 - \frac{41}{64} e^4$	$4 e - \frac{41}{16} e^3$	1
	$-\frac{e}{2} + e^3$	$-\frac{1}{2} + 3 e^2$	2
	$-\frac{3}{8} e^2 + \frac{11}{16} e^4$	$-\frac{3}{4} e + \frac{11}{4} e^3$	3
	$-\frac{7}{24} e^3$	$-\frac{7}{8} e^2$	4
	$-\frac{95}{384} e^4$	$-\frac{95}{96} e^3$	5

LOW ECCENTRICITY

TABLE 8

$q = 1, \quad q' = 3$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{15}{8} c'^6$	$c^2 \gamma^4$	$2 \gamma^3 (2 - 3 \gamma^2)$	- 3
$- \frac{15}{4} c'^5 \gamma'$	$c \gamma^3 (2 - 3 \gamma^2)$	$c^{-1} \gamma^2 (6 - 23 \gamma^2 + 18 \gamma^4)$	- 2
$\frac{15}{8} c'^4 \gamma'^2$	$\gamma^2 (6 - 20 \gamma^2 + 15 \gamma^4)$	$2 \gamma (6 - 40 \gamma^2 + 45 \gamma^4)$	- 1
$- \frac{15}{2} c'^3 \gamma'^3$	$c \gamma (1 - 5 \gamma^2 + 5 \gamma^4)$	$c^{-1} (1 - 17 \gamma^2 + 45 \gamma^4 - 30 \gamma^6)$	0
$\frac{15}{8} c'^2 \gamma'^4$	$c^2 (1 - 10 \gamma^2 + 15 \gamma^4)$	$- 2 \gamma (11 - 50 \gamma^2 + 45 \gamma^4)$	1
$\frac{15}{4} c' \gamma'^5$	$c^3 \gamma (1 - 3 \gamma^2)$	$c (1 - 13 \gamma^2 + 18 \gamma^4)$	2
$\frac{15}{8} \gamma'^6$	$c^4 \gamma^2$	$2 c^2 \gamma (1 - 3 \gamma^2)$	3
	$B_{i'q'}$		i'
	$\frac{1}{8} e'^2$		1
	$- e'$		2
	$1 - 6 e'^2$		3
	$5 e'$		4
	$\frac{127}{8} e'^2$		5
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$\frac{7}{128} e^4$	$\frac{7}{32} e^3$	- 3
	$\frac{1}{6} e^3$	$\frac{1}{2} e^2$	- 2
	$\frac{11}{8} e^2 + \frac{7}{48} e^4$	$\frac{11}{4} e + \frac{7}{12} e^3$	- 1
	$- \frac{5}{2} e - \frac{15}{8} e^3$	$- \frac{5}{2} - \frac{45}{8} e^2$	0
	$1 + 2 e^2 - \frac{41}{64} e^4$	$4 e - \frac{41}{16} e^3$	1
	$- \frac{e}{2} + e^3$	$- \frac{1}{2} + 3 e^2$	2
	$- \frac{3}{8} e^2 + \frac{11}{16} e^4$	$- \frac{3}{4} e + \frac{11}{4} e^3$	3
	$- \frac{7}{24} e^3$	$- \frac{7}{8} e^2$	4
	$- \frac{95}{384} e^4$	$- \frac{95}{96} e^3$	5

LOW ECCENTRICITY

TABLE 9

$q = 1, \; q' = -3$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{15}{8}\gamma'^6$	$c^2\gamma^4$	$2\gamma^3(2 - 3\gamma^2)$	-3
$\frac{15}{4}c'\gamma'^5$	$c\gamma^3(2 - 3\gamma^2)$	$c^{-1}\gamma^2(6 - 23\gamma^2 + 18\gamma^4)$	-2
$\frac{15}{8}c'^2\gamma'^4$	$\gamma^2(6 - 20\gamma^2 + 15\gamma^4)$	$2\gamma(6 - 40\gamma^2 + 45\gamma^4)$	-1
$\frac{15}{2}c'^3\gamma'^3$	$c\gamma(1 - 5\gamma^2 + 5\gamma^4)$	$c^{-1}(1 - 17\gamma^2 + 45\gamma^4 - 30\gamma^6)$	0
$\frac{15}{8}c'^4\gamma'^2$	$c^2(1 - 10\gamma^2 + 15\gamma^4)$	$-2\gamma(11 - 50\gamma^2 + 45\gamma^4)$	1
$-\frac{15}{4}c'^5\gamma'$	$c^3\gamma(1 - 3\gamma^2)$	$c(1 - 13\gamma^2 + 18\gamma^4)$	2
$\frac{15}{8}c'^6$	$c^4\gamma^2$	$2c^2\gamma(1 - 3\gamma^2)$	3
	$B_{i'q'}$		i'
	$\frac{127}{8}e'^2$		-5
	$5e'$		-4
	$1 - 6e'^2$		-3
	$-e'$		-2
	$\frac{1}{8}e'^2$		-1
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$\frac{7}{128}e^4$	$\frac{7}{32}e^3$	-3
	$\frac{1}{6}e^3$	$\frac{1}{2}e^2$	-2
	$\frac{11}{8}e^2 + \frac{7}{48}e^4$	$\frac{11}{4}e + \frac{7}{12}e^3$	-1
	$-\frac{5}{2}e - \frac{15}{8}e^3$	$-\frac{5}{2} - \frac{45}{8}e^2$	0
	$1 + 2e^2 - \frac{41}{64}e^4$	$4e - \frac{41}{16}e^3$	1
	$-\frac{e}{2} + e^3$	$-\frac{1}{2} + 3e^2$	2
	$-\frac{3}{8}e^2 + \frac{11}{16}e^4$	$-\frac{3}{4}e + \frac{11}{4}e^3$	3
	$-\frac{7}{24}e^3$	$-\frac{7}{8}e^2$	4
	$-\frac{95}{384}e^4$	$-\frac{95}{96}e^3$	5

LOW ECCENTRICITY

TABLE 10

$q = 3,\ q' = 1$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{15}{8} c'^4 \gamma'^2$	γ^6	$6 \gamma^5$	-3
$\frac{15}{4} c'^3 \gamma' (1 - 3 \gamma'^2)$	$c \gamma^5$	$c^{-1} \gamma^4 (5 - 6 \gamma^2)$	-2
$\frac{15}{8} c'^2 (1 - 10 \gamma'^2 + 15 \gamma'^4)$	$c^2 \gamma^4$	$2 \gamma^3 (2 - 3 \gamma^2)$	-1
$-\frac{15}{2} c' \gamma' (1 - 5 \gamma'^2 + 5 \gamma'^4)$	$c^3 \gamma^3$	$3 c \gamma^2 (1 - 2 \gamma^2)$	0
$\frac{15}{8} \gamma'^2 (6 - 20 \gamma'^2 + 15 \gamma'^4)$	$c^4 \gamma^2$	$2 c^2 \gamma (1 - 3 \gamma^2)$	1
$-\frac{15}{4} c' \gamma'^3 (2 - 3 \gamma'^2)$	$c^5 \gamma$	$c^3 (1 - 6 \gamma^2)$	2
$\frac{15}{8} c'^2 \gamma'^4$	c^6	$-6 c^4 \gamma$	3
	$B_{i'q'}$		i'
	$\frac{11}{8} e'^2$		-1
	e'		0^*
	$1 + 2 e'^2$		1
	$3 e'$		2
	$\frac{53}{8} e'^2$		3
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$\frac{75}{128} e^4$	$\frac{75}{32} e^3$	-1
	$-\frac{35}{8} e^3$	$-\frac{105}{8} e^2$	0^*
	$\frac{57}{8} e^2 - \frac{65}{16} e^4$	$\frac{57}{4} e - \frac{65}{4} e^3$	1
	$-\frac{9}{2} e + \frac{33}{4} e^3$	$-\frac{9}{2} + \frac{99}{4} e^2$	2
	$1 - 6 e^2 + \frac{591}{64} e^4$	$-12 e + \frac{591}{16} e^3$	3
	$\frac{3}{2} e - \frac{57}{8} e^3$	$\frac{3}{2} - \frac{171}{8} e^2$	4
	$\frac{15}{8} e^2 - \frac{135}{16} e^4$	$\frac{15}{4} e - \frac{135}{4} e^3$	5
	$\frac{9}{4} e^3$	$\frac{27}{4} e^2$	6
	$\frac{343}{128} e^4$	$\frac{343}{32} e^3$	7

TABLE 11

$q = 3,\ q' = -1$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{15}{8} e'^2 \gamma'^4$	γ^6	$6\gamma^5$	-3
$\frac{15}{4} e' \gamma'^3 (2 - 3\gamma'^2)$	$e\gamma^5$	$e^{-1} \gamma^4 (5 - 6\gamma^2)$	-2
$\frac{15}{8} \gamma'^2 (6 - 20\gamma'^2 + 15\gamma'^4)$	$e^2 \gamma^4$	$2\gamma^3 (2 - 3\gamma^2)$	-1
$\frac{15}{2} e' \gamma' (1 - 5\gamma'^2 + 5\gamma'^4)$	$e^3 \gamma^3$	$3 e \gamma^2 (1 - 2\gamma^2)$	0
$\frac{15}{8} e'^2 (1 - 10\gamma'^2 + 15\gamma'^4)$	$e^4 \gamma^2$	$2 e^2 \gamma (1 - 3\gamma^2)$	1
$-\frac{15}{4} e'^3 \gamma' (1' - 3\gamma'^2)$	$e^5 \gamma$	$e^3 (1 - 6\gamma^2)$	2
$\frac{15}{8} e'^4 \gamma'^2$	e^6	$-6 e^4 \gamma$	3
	$B_{i'q'}$		i'
	$\frac{53}{8} e'^2$		-3
	$3 e'$		-2
	$1 + 2 e'^2$		-1
	e'		0^*
	$\frac{11}{8} e'^2$		1
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$\frac{75}{128} e^4$	$\frac{75}{32} e^3$	-1
	$-\frac{35}{8} e^3$	$-\frac{105}{8} e^2$	0^*
	$\frac{57}{8} e^2 - \frac{65}{16} e^4$	$\frac{57}{4} e - \frac{65}{4} e^3$	1
	$-\frac{9}{2} e + \frac{33}{4} e^3$	$-\frac{9}{2} + \frac{99}{4} e^2$	2
	$1 - 6 e^2 + \frac{591}{64} e^4$	$-12 e + \frac{591}{16} e^3$	3
	$\frac{3}{2} e - \frac{57}{8} e^3$	$\frac{3}{2} - \frac{171}{8} e^2$	4
	$\frac{15}{8} e^2 - \frac{135}{16} e^4$	$\frac{15}{4} e - \frac{135}{4} e^3$	5
	$\frac{9}{4} e^3$	$\frac{27}{4} e^2$	6
	$\frac{343}{128} e^4$	$\frac{343}{32} e^3$	7

LOW ECCENTRICITY

TABLE 12

$q = 3, \ q' = 3$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{5}{8} c'^6$	γ^6	$6 \gamma^5$	-3
$-\frac{15}{4} c'^5 \gamma'$	$c \gamma^5$	$c^{-1} \gamma^4 (5 - 6 \gamma^2)$	-2
$\frac{75}{8} c'^4 \gamma'^2$	$c^2 \gamma^4$	$2 \gamma^3 (2 - 3 \gamma^2)$	-1
$-\frac{25}{2} c'^3 \gamma'^3$	$c^3 \gamma^3$	$3 c \gamma^2 (1 - 2 \gamma^2)$	0
$\frac{75}{8} c'^2 \gamma'^4$	$c^4 \gamma^2$	$2 c^2 \gamma (1 - 3 \gamma^2)$	1
$-\frac{15}{4} c' \gamma'^5$	$c^5 \gamma$	$c^3 (1 - 6 \gamma^2)$	2
$\frac{5}{8} \gamma'^6$	c^6	$-6 c^4 \gamma$	3
	$B_{i'q'}$		i'
	$\frac{1}{8} e'^2$		1
	$-e'$		2
	$1 - 6 e'^2$		3
	$5 e'$		4
	$\frac{127}{8} e'^2$		5
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$\frac{75}{128} e^4$	$\frac{75}{32} e^3$	-1
	$-\frac{35}{8} e^3$	$-\frac{105}{8} e^2$	0
	$\frac{57}{8} e^2 - \frac{65}{16} e^4$	$\frac{57}{4} e - \frac{65}{4} e^3$	1
	$-\frac{9}{2} e + \frac{33}{4} e^3$	$-\frac{9}{2} + \frac{99}{4} e^2$	2
	$1 - 6 e^2 + \frac{591}{64} e^4$	$-12 e + \frac{591}{16} e^3$	3
	$\frac{3}{2} e - \frac{57}{8} e^3$	$\frac{3}{2} - \frac{171}{8} e^2$	4
	$\frac{15}{8} e^2 - \frac{135}{16} e^4$	$\frac{15}{4} e - \frac{135}{4} e^3$	5
	$\frac{9}{4} e^3$	$\frac{27}{4} e^2$	6
	$\frac{343}{128} e^4$	$\frac{343}{32} e^3$	7

TABLE 13

$q = 3,\ q' = -3$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{5}{8}\gamma'^6$	γ^6	$6\gamma^5$	-3
$\frac{15}{4}c'\gamma'^5$	$c\gamma^5$	$c^{-1}\gamma^4(5-6\gamma^2)$	-2
$\frac{75}{8}c'^2\gamma'^4$	$c^2\gamma^4$	$2\gamma^3(2-3\gamma^2)$	-1
$\frac{25}{2}c'^3\gamma'^3$	$c^3\gamma^3$	$3c\gamma^2(1-2\gamma^2)$	0
$\frac{75}{8}c'^4\gamma'^2$	$c^4\gamma^2$	$2c^2\gamma(1-3\gamma^2)$	1
$\frac{15}{4}c'^5\gamma'$	$c^5\gamma$	$c^3(1-6\gamma^2)$	2
$\frac{5}{8}c'^6$	c^6	$-6c^4\gamma$	3
	$B_{i'q'}$		i'
	$\frac{127}{8}e'^2$		-5
	$5e'$		-4
	$1-6e'^2$		-3
	$-e'$		-2
	$\frac{1}{8}e'^2$		-1
	C_{iq}	$\frac{dC_{iq}}{de}$	i
	$\frac{75}{128}e^4$	$\frac{75}{32}e^3$	-1
	$-\frac{35}{8}e^3$	$-\frac{105}{8}e^2$	0
	$\frac{57}{8}e^2-\frac{65}{16}e^4$	$\frac{57}{4}e-\frac{65}{4}e^3$	1
	$-\frac{9}{2}e+\frac{33}{4}e^3$	$-\frac{9}{2}+\frac{99}{4}e^2$	2
	$1-6e^2+\frac{591}{64}e^4$	$-12e+\frac{591}{16}e^3$	3
	$\frac{3}{2}e-\frac{57}{8}e^3$	$\frac{3}{2}-\frac{171}{8}e^2$	4
	$\frac{15}{8}e^2-\frac{135}{16}e^4$	$\frac{15}{4}e-\frac{135}{4}e^3$	5
	$\frac{9}{4}e^3$	$\frac{27}{4}e^2$	6
	$\frac{343}{128}e^4$	$\frac{343}{32}e^3$	7

THE DISTURBING FUNCTION FOR ARBITRARY ECCENTRICITY

By means of formulas in Reference 4, (9) may be expressed in the following form

$$\frac{k^2 m'}{a'^{p'}} a^p A_{\nu q} A'_{\nu q'} B_{i'q'} \sum_i C_{iq} \cos (i E + i' \ell' + q\omega + q'\omega' + \nu\theta) \tag{18}$$

where C_{iq} is a function of β and where

$$\beta = \frac{1}{e} (1 - \sqrt{1 - e^2}) = \frac{e}{1 + \sqrt{1 - e^2}} \tag{19}$$

The quantity β is thus of the same order as e, the eccentricity of the satellite's orbit. Further, in C_{iq} the expansions in β are in terms of closed polynomials rather than infinite series as would be the case if the expansion had been in terms of the mean anomaly of the satellite instead of the eccentric anomaly, E.

Thus the expansion of R, or even $(r/a)R$ which as shown below is useful in deriving the determining function, can be expressed for given values of q and q' in the form

$$\frac{r}{a} R = \frac{k^2 m'}{a'^{p'}} a^p \sum_\nu \sum_i \sum_{i'} A'_{\nu q'} A_{\nu q} B_{i'q'} C_{iq} \cos (i E + i'\ell' + \nu\theta + q\omega + q'\omega') \tag{20}$$

The coefficients $A'_{\nu q'}$, $A_{\nu q}$ $B_{i'q'}$, and C_{iq} are given in Tables 1 through 13 for arbitrary eccentricity.

The expansion of $(r/a)R$ is used in finding the determining function which is now discussed.

THE DETERMINING FUNCTION AND PERTURBATIONS FOR ARBITRARY ECCENTRICITY

The determining function for the arbitrary eccentricity case is obtained in a slightly different fashion from that of low eccentricity. The development for this case is the same through Equation (12) however. The difference in the form of the determining function is due to the need of using eccentric anomaly instead of mean anomaly in order to obtain closed expressions involving eccentricity.

Since in this section the eccentric anomaly of the satellite rather than the mean anomaly is used, Equation (12) takes the form

$$\frac{\partial Q}{\partial E} + m \frac{r}{a} \frac{\partial Q}{\partial \ell'} = \frac{r}{na} R_p \tag{21}$$

where

$$m = \frac{n'}{n} = \frac{m_{\oplus} + M'}{m_{\oplus}} \left(\frac{a}{a'} \right)^{3/2}$$

and where $m_{\oplus}$ and M' are the mass of the earth and disturbing body respectively.

But it has been shown that a typical term in the expansion of $(r/a)R_p$ is of the form

$$A \cos (iE + i'\ell' + \phi).$$

Hence we may write Equation (15) in the form

$$\frac{\partial Q}{\partial E} + m \frac{\partial Q}{\partial \ell'} = \frac{A}{n} \cos (iE + i'\ell' + \phi) + m_1 \frac{\partial Q}{\partial \ell'} \cos E \tag{22}$$

where $m_1 = me$. Equation (22) is a type considered by Hansen, Reference 6, whose solution may be represented in the form

$$Q = Q_0 + m_1 Q_1 + m_1^2 Q_2 + \cdots \tag{23}$$

Hansen's method for finding the expansion represented by Equation (23) has been described in Reference 2, which we repeat for the convenience of the reader.

We first neglect the terms of order m_1 and higher in Equations (22) and (23) to obtain the equation

$$\frac{\partial Q_0}{\partial E} + m \frac{\partial Q_0}{\partial \ell'} = \frac{A}{n} \cos (iE + i'\ell' + \phi) \tag{24}$$

The solution of Equation (24) is given by

$$Q_0 = \frac{A \sin (iE + i'\ell' + \phi)}{n(i + i'm)} \tag{25}$$

which may be verified by noting that Equation (25) satisfied Equation (24).

The next step is to neglect terms of order m_1^2 and write

$$Q = \frac{A \sin (iE + i'\ell' + \phi)}{n(i + i'm)} + m_1 Q_1 \tag{26}$$

which is substituted into Equation (22) to obtain

$$\frac{\partial Q_1}{\partial E} + m \frac{\partial Q_1}{\partial \ell'} = \frac{\partial Q_0}{\partial \ell'} \cos E \tag{27}$$

and in general we may write

$$\frac{\partial Q_j}{\partial E} + m \frac{\partial Q_j}{\partial \ell'} = \frac{\partial Q_{j-i}}{\partial \ell'} \cos E \tag{28}$$

Then, from Equations (23), (25), and (27) the solution for Q may be found to order m^2

$$Q = \frac{A}{n(i + i'm)} (Q_0' + me Q_1' + m^2 e^2 Q_2') \tag{29}$$

where Q_0', Q_1' and Q_2' are trigonometric functions given in Table 14.

Substituting the value of Q from Equation (29) into Equation (13) we find that

$$\delta L = \left(\frac{\partial Q_0'}{\partial E} + me \frac{\partial Q_1'}{\partial E} + m^2 e^2 \frac{\partial Q_2'}{\partial E}\right) \frac{a}{r} \frac{A}{ni + n'i'}$$

$$\delta G = \left(\frac{\partial Q_0'}{\partial \omega} + me \frac{\partial Q_1'}{\partial \omega} + m^2 e^2 \frac{\partial Q_2'}{\partial \omega}\right) \frac{A}{ni + n'i'} \tag{30}$$

$$\delta H = \left(\frac{\partial Q_0'}{\partial \theta} + me \frac{\partial Q_1'}{\partial \theta} + m^2 e^2 \frac{\partial Q_2'}{\partial \theta}\right) \frac{A}{ni + n'i'}$$

The differential coefficients with respect to E, ω, and θ are listed in Table 14. Similarly

$$\delta \ell = - \frac{\partial Q}{\partial L} = - \frac{2L}{\mu} \frac{\partial Q}{\partial a} + \frac{G^2}{eL^3} \frac{\partial Q}{\partial e}$$

$$\delta \omega = - \frac{\partial Q}{\partial G} = \frac{G}{eL^2} \frac{\partial Q}{\partial e} + \frac{\cos I}{G} \frac{\partial Q}{\partial (\cos I)} = \frac{G}{eL^2} \frac{\partial Q}{\partial e} - \frac{(1-2\gamma^2)}{4G\gamma} \frac{\partial Q}{\partial \gamma} \tag{31}$$

$$\delta \Omega = - \frac{\partial Q}{\partial H} = - \frac{1}{G} \frac{\partial Q}{\partial (\cos I)} = \frac{1}{4G\gamma} \frac{\partial Q}{\partial \gamma} .$$

We also need the formulas

$$\frac{\partial E}{\partial e} = \frac{a}{r} \sin E$$

$$\frac{d\beta}{de} = \frac{(1 + \beta^2)^2}{2(1-\beta^2)} \tag{32}$$

$$\frac{dn}{da} = - \frac{3}{2} \frac{n}{a}$$

For group 1 of Equation (5), the quantity, A, appearing in Equation (29) is given in Tables 1-13 for Arbitrary Eccentricity. It is of the form

$$\frac{k^2 m'}{a'^3} \quad a^2 A'_{\nu q'} \quad A_{\nu q} \quad A_{\nu q} \quad B_{i' q'} \quad C_{iq}$$

so that $\partial Q / \partial a$ is given by

$$\frac{\partial Q}{\partial a} = \frac{k^2 m' a (7ni + 4n'i')}{2a'^3 (ni + n'i')^2} \quad A'_{\nu q'} \, A_{\nu q} \, B_{i'q'} \, C_{iq} \quad [Q_0' + meQ_1' + m^2 e^2 Q_2']$$

$$+ \frac{A}{ni + n'i'} \left[\frac{\partial meQ_1'}{\partial a} + \frac{\partial m^2 e^2 Q_2'}{\partial a} \right] \tag{33}$$

where $\partial meQ_1'/\partial a$ and $\partial m^2 e^2 Q_2'/\partial a$ are given in Table 14. Similarly, for group 2 of Equation (5) we find

$$\frac{\partial Q}{\partial a} = \frac{3}{2} \frac{k^2 m' a^2}{a'^4} \frac{(3ni + 2n'i')}{(ni + n'i')^2} \quad A'_{\nu q'} \, A_{\nu q} \, B_{i'q'} \, C_{iq} \, [Q_0' + meQ_1' \, m^2 e^2 Q_2']$$

$$+ \frac{A}{ni + n'i'} \left[\frac{\partial meQ_1'}{\partial a} + \frac{\partial m^2 e^2 Q_2'}{\partial a} \right] \tag{34}$$

For group 1, $\partial Q/\partial e$ is given by the formula

$$\frac{\partial Q}{\partial e} = \frac{k^2 m' a^2}{a'^3} \left\{ \frac{(1+\beta^2)^2}{2(1-\beta^2)} \quad A'_{\nu q'} \, A_{\nu q} \, B_{i'q'} \frac{\partial C_{iq}}{\partial \beta} \, [Q_0' + meQ_1' + m^2 e^2 Q_2'] \right.$$

$$+ \frac{A}{ni + n'i'} \left[\frac{\partial Q_0'}{\partial E} \frac{a}{r} \sin E + mQ_1' + me \frac{\partial Q_1'}{\partial E} \frac{a}{r} \sin E + 2m^2 eQ_2' \right.$$

$$\left. \left. + m^2 e^2 \frac{\partial Q_2'}{\partial E} \frac{a}{r} \sin E \right] \right\} \tag{35}$$

For group 2, the factor $k^2 m' a^2/a'^3$ is replaced by $k^2 m' a^3/a'^4$. Finally for group 1, $\partial Q/\partial \gamma$ is given by

$$\frac{\partial Q}{\partial \gamma} = \frac{k^2 m' a^2}{a'^3} \, A'_{\nu q'} \, \frac{\partial A_{\nu q}}{\partial \gamma} \, B_{i'q'} \, C_{iq} \quad [Q_0' + meQ_1' + m^2 e^2 Q_2'] \tag{36}$$

The corresponding derivative for $\partial Q/\partial \gamma$ for group 2 is again obtained by replacing the factor $k^2 m' a^2/a'^3$ by $k^2 m' a^3/a'^4$ in Equation (36).

SPECIAL CASES

Several interesting special cases arise which simplify the calculation of certain terms in the determining function, they are, (i.) $i \neq 0$, $i' = 0$, (ii.) $i = 0$ $i' \neq 0$ and (iii.) $i = i' = 0$.

(i.) The Case $i \neq 0$, $i' = 0$

The differential equation for Q, Equation (21), then becomes

$$\frac{\partial Q}{\partial E} = \frac{1}{n} \frac{r}{a} R_p \tag{37}$$

Hence Q is of the form

$$Q = \frac{k^2 m'}{n a'^{p'}} a^p A'_{\nu q'} A_{\nu q} B_{0q'} C_{iq} \sin(iE + \phi)$$

So that $Q = Q_0$ with $i' = 0$ and thus agrees with the result of the general case.

(ii.) The Case $i = 0$, $i' \neq 0$

The differential equation for Q, Equation (21), for this case becomes

$$n' \frac{\partial Q}{\partial \ell'} = R_p \tag{38}$$

Now, from Equation (9) we find Q is of the form

$$Q = \frac{k^2 m' a^p}{n' a'^{p'}} A'_{\nu q'} A_{\nu q} B_{i'q'} C_{0q} \sin(i' \ell' + \phi) \tag{39}$$

For this case we find that

$$Q = Q_0$$

with the subscript i set equal to zero.

(iii.) The Case $i = i' = 0$

Let R_s be the terms of R that are independent of both i and i′ obtained by averaging R over the mean anomalies of the satellite and the disturbing body. Then Equation (21) may be written as

$$\frac{\partial Q}{\partial E} = \frac{r}{na} (R - R_s) \tag{40}$$

Integrating we find

$$Q = \frac{R_s}{n} (E - \ell), \tag{41}$$

since $$d\ell = \frac{r}{a} dE \text{ and } \frac{1}{2\pi}\int_0^{2\pi} \frac{r}{a} R dE = R_s .$$

Thus, $(r/a)R = R_s$ for those terms in $(r/a)R$ for which $i = i' = 0$. Thus

$$R_s = \frac{k^2 m' a^2}{a'^3} \sum A'_{\nu q'} \; A_{\nu q} \; B_{0q'} \; C_{0q} \; \cos (q\omega + q'\omega' + \nu\theta)$$

$$+ \frac{k^2 m' a^3}{a'^4} \sum A'_{\nu q'} \; A_{\nu q} \; B_{0q'} \; C_{0q} \; \cos (q\omega + q'\omega' + \nu\theta) \tag{42}$$

where the summations are over q, q′ and ν. We have

$$\delta L = \frac{\partial Q}{\partial \ell} = \frac{R_s}{n}\left(\frac{a}{r} - 1\right)$$

$$\delta G = \frac{q}{n}\,\bar{R}_s \tag{43}$$

$$\delta H = \frac{\nu}{n}\,\bar{R}_s$$

where $\bar{R}_s$ is the expression obtained when the cosine terms of R are changed to sine. The perturbations in ℓ, ω, and Ω are given by Equations (21). The expressions for $\partial Q/\partial a$, $\partial Q/\partial e$, and $\partial Q/\partial \gamma$ are

$$\frac{\partial Q}{\partial a} = \frac{7\,a\,k^2\,m'}{2\,n\,a'^3}\,(E - \ell) \sum A'_{\nu q'}\, A_{\nu q}\, B_{0q'}\, C_{0q} \quad \cos\,(q\,\omega + q'\,\omega' + \nu\,\theta)$$

$$+ \frac{9\,a^2\,k^2\,m'}{2\,n\,a'^4}\,(E - \ell) \sum A'_{\nu q'}\, A_{\nu q}\, B_{0q'}\, C_{0q} \quad \cos\,(q\,\omega + q'\,\omega' + \nu\,\theta) \tag{44}$$

$$\frac{\partial Q}{\partial e} = \frac{k^2\,m'\,a^2}{n\,a'^3}\,\frac{a}{r}\sin E \sum A'_{\nu q'}\, A_{\nu q}\, B_{0q'}\, C_{0q} \quad \cos\,(q\,\omega + q'\,\omega' + \nu\,\theta)$$

$$+ \frac{k^2\,m'\,a^3}{n\,a'^4}\,\frac{a}{r}\sin E \sum A'_{\nu q'}\, A_{\nu q}\, B_{0q'}\, C_{0q} \quad \cos\,(q\,\omega + q'\,\omega' + \nu\,\theta)$$

$$+ \frac{k^2\,m'\,a^2}{2\,n\,a'^3}\,\frac{(1+\beta^2)^2}{1-\beta^2}\,(E - \ell) \sum A'_{\nu q'}\, A_{\nu q}\, B_{0q'}\, \frac{\partial C_{0q}}{\partial \beta} \quad \cos\,(q\,\omega + q'\,\omega' + \nu\,\theta)$$

$$+\frac{k^2 m' a^3}{2 n a'^4} \frac{(1+\beta^2)^2}{1-\beta^2} (E - \ell) \sum A'_{\nu q'} A_{\nu q} B_{0q'} \frac{\partial C_{0q}}{\partial \beta} \cos (q\omega + q'\omega' + \nu\theta) \tag{45}$$

$$\frac{\partial Q}{\partial \gamma} = \frac{k^2 m' a^2}{n a'^3} (E - \ell) \sum A'_{\nu q'} \frac{\partial A_{\nu q}}{\partial \gamma} B_{0q'} C_{0q} \cos (q\omega + q'\omega' + \nu\theta)$$

$$+\frac{k^2 m' a^3}{n a'^4} (E - \ell) \sum A'_{\nu q'} \frac{\partial A_{\nu q}}{\partial \gamma} B_{0q'} C_{0q} \cos (q\omega + q'\omega' + \nu\theta) \tag{46}$$

ILLUSTRATIVE EXAMPLE OF THE USE OF THE TABLES FOR ARBITRARY ECCENTRICITY

In order to illustrate the use of the tables, the contribution to the perturbation in ω resulting from the term defined by the indices $q = 2$, $q' = -2$, $\nu = -1$, $i' = -2$, and $i = 3$ is obtained.

It is assumed that the "mean" elements a, e, i, ω and Ω are known. Then, for a given value of ℓ, E may be determined from Kepler's equation by using the mean eccentricity. Then r/a is known also as are the elements of the disturbing body.

1. From Table 14 we find

$$Q_0' = \sin (3E - 2\ell' + \phi) \text{ where } \phi = 2\omega - 2\omega' - \theta$$

$$Q_1' = -\frac{\sin (4E - 2\ell' + \phi)}{2 - m} - \frac{1}{2} \frac{\sin (2E - 2\ell' + \phi)}{1 - m}$$

$$Q_2' = \frac{\sin (E - 2\ell' + \phi)}{2(1-m)(1-2m)} + \frac{\sin (3E - 2\ell' + \phi)}{2(1-m)(2-m)} + \frac{\sin (5E - 2\ell' + \phi)}{2(2-m)(5-2m)}$$

$$\frac{\partial Q_0'}{\partial E} = 3 \cos (3E - 2\ell' + \phi)$$

$$\frac{\partial Q_1'}{\partial E} = \frac{-2}{2-m} \cos(4E - 2\ell' + \phi) - \frac{1}{1-m} \cos(2E - 2\ell' + \phi)$$

$$\frac{\partial Q_2'}{\partial E} = \frac{\cos(E - 2\ell' + \phi)}{2(1-2m)(1-m)} - \frac{\cos(3E - 2\ell' + \phi)}{(1-2m)(2-m)} + \frac{5}{2} \frac{\cos(5E - 2\ell' + \phi)}{(2-m)(5-2m)}$$

2. From Table 2 we find

$$A'_{-1-2} = 3c'\gamma'^3$$

$$B_{3-2} = 1 - \frac{5}{2} e'^2$$

$$A_{-12} = c\gamma^3$$

$$C_{32} = \frac{-\beta}{(1+\beta^2)^3}$$

Then

$$A = \frac{k^2 m' a^2}{a'^3} A_{-1-2} A_{-12} B_{3-2} C_{32}$$

Also,

$$\frac{dC_{32}}{d\beta} = \frac{-1 + 5\beta^2}{(1+\beta^2)^4}$$

We then find dQ/de from Equation (35). From Table 2 again we have

$$\frac{dA_{-12}}{d\gamma} = \frac{\gamma^2}{c}(3 - 4\gamma^2)$$

Hence, we can calculate $dQ/d\gamma$ by means of Equation (36).

Finally, having calculated dQ/de and $dQ/d\gamma$ we may derive the perturbation in ω by means of Equation (31).

ARBITRARY ECCENTRICITY

TABLE 1

$q = 0, \quad q' = 0$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{1}{4}(1 - 6\gamma'^2 + 6\gamma'^4)$	$1 - 6\gamma^2 + 6\gamma^4$	$-12\gamma(1 - 2\gamma^2)$	0
$3c'\gamma'(1 - 2\gamma'^2)$	$c\gamma(1 - 2\gamma^2)$	$c^{-1}(1 - 8\gamma^2 + 8\gamma^4)$	1
$3c'^2\gamma'^2$	$c^2\gamma^2$	$2\gamma(1 - 2\gamma^2$	2
	$B_{i'q'}$		i'
	$\frac{9}{4}e'^2$		-2
	$\frac{3}{2}e'$		-1
	$(1 - e'^2)^{-3/2}$		0*
	$\frac{3}{2}e'$		1
	$\frac{9}{4}e'^2$		2
	$C_{iq} \quad (1 + \beta^2)^{-3}$	$\frac{dC_{iq}}{d\beta} \quad (1 + \beta^2)^{-4}$	i
	$-\beta^3$	$-3\beta^2(1 - \beta^2)$	-3
	$3\beta^2(1 + \beta^2)$	$6\beta(1 - \beta^4)$	-2
	$-3\beta(1 + 3\beta^2 + \beta^4)$	$-3(1 - \beta^2)(1 + 5\beta^2 + \beta^4)$	-1
	$1 + 9\beta^2 + 9\beta^4 + \beta^6$	$12\beta(1 - \beta^4)$	0*
	$-3\beta(1 + 3\beta^2 + \beta^4)$	$-3(1 - \beta^2)(1 + 5\beta^2 + \beta^4)$	1
	$3\beta^2(1 + \beta^2)$	$6\beta(1 - \beta^4)$	2
	$-\beta^3$	$-3\beta^2(1 - \beta^2)$	3

ARBITRARY ECCENTRICITY

TABLE 2

$q = 2, \quad q' = -2$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{3}{4}\gamma'^4$	γ^4	$4\gamma^3$	-2
$3e'\gamma'^3$	$e\gamma^3$	$e^{-1}\gamma^2(3-4\gamma^2)$	-1
$\frac{9}{2}e'^2\gamma'^2$	$e^2\gamma^2$	$2\gamma(1-2\gamma^2)$	0
$3e'^3\gamma'$	$e^3\gamma$	$e(1-4\gamma^2)$	1
$\frac{3}{4}e'^4$	e^4	$-4e^2\gamma$	2
	$B_{i'q'}$		i'
	$\frac{17}{2}e'^2$		-4
	$\frac{7}{2}e'$		-3
	$1-\frac{5}{2}e'^2$		-2
	$-\frac{1}{2}e'$		-1
	$C_{iq} \quad (1+\beta^2)^{-3}$	$\frac{dC_{iq}}{d\beta} \quad (1+\beta^2)^{-4}$	i
	$-\beta^5$	$-5\beta^4+\beta^6$	-3
	$\beta^4(5+\beta^2)$	$4\beta^3(5-\beta^2)$	-2
	$-\beta^3(10+5\beta^2)$	$5\beta^2(-2+\beta^2)(3+\beta^2)$	-1
	$10\beta^2(1+\beta^2)$	$20\beta(1-\beta^4)$	0
	$-5\beta(1+2\beta^2)$	$-5(1-2\beta^2)(1+3\beta^2)$	1
	$1+5\beta^2$	$4\beta(1-5\beta^2)$	2
	$-\beta$	$-1+5\beta^2$	3

ARBITRARY ECCENTRICITY

TABLE 3

$q = 2, \; q' = 0$

$A'_{\nu q'}$	$A_{\nu q}$	$\dfrac{dA_{\nu q}}{d\gamma}$	ν
$\frac{3}{2} c'^2 \gamma'^2$	γ^4	$4\gamma^3$	-2
$3c'\gamma'(1-2\gamma'^2)$	$c\gamma^3$	$c^{-1}\gamma^2(3-4\gamma^2)$	-1
$\frac{3}{2}(1-6\gamma'^2+6\gamma'^4)$	$c^2\gamma^2$	$2\gamma(1-2\gamma^2)$	0
$-3c'\gamma'(1-2\gamma'^2)$	$c^3\gamma$	$c(1-4\gamma^2)$	1
$\frac{3}{2} c'^2 \gamma'^2$	c^4	$-4c^2\gamma$	2
	$B_{i'q'}$		i'
	$\frac{9}{4} e'^2$		-2
	$\frac{3}{2} e'$		-1
	$(1-e^2)^{-3/2}$		0*
	$\frac{3}{2} e'$		1
	$\frac{9}{4} e'^2$		2
	$C_{iq} \; (1+\beta^2)^{-3}$	$\dfrac{dC_{iq}}{d\gamma} \; (1+\beta^2)^{-4}$	i
	$-\beta^5$	$-5\beta^4+\beta^6$	-3
	$\beta^4(5+\beta^2)$	$4\beta^3(5-\beta^2)$	-2
	$-\beta^3(10+5\beta^2)$	$5\beta^2(-2+\beta^2)(3+\beta^2)$	-1
	$10\beta^2(1+\beta^2)$	$20\beta(1-\beta^4)$	0*
	$-5\beta(1+2\beta^2)$	$-5(1-2\beta^2)(1+3\beta^2)$	1
	$1+5\beta^2$	$4\beta(1-5\beta^2)$	2
	$-\beta$	$-1+5\beta^2$	3

ARBITRARY ECCENTRICITY

TABLE 4

$q = 0, \ q' = 2$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{3}{2} c'^4$	$c^2 \gamma^2$	$2 \gamma (1 - 2 \gamma^2)$	- 2
$- 3 c'^3 \gamma'$	$c \gamma (1 - 2 \gamma^2)$	$c^{-1} (1 - 8 \gamma^2 + 8 \gamma^4)$	- 1
$\frac{3}{2} c'^2 \gamma'^2$	$1 - 6 \gamma^2 + 6 \gamma^4$	$- 12 \gamma (1 - 2 \gamma^2)$	0
$3 c' \gamma'^3$	$c \gamma (1 - 2 \gamma^2)$	$c^{-1} (1 - 8 \gamma^2 + 8 \gamma^4)$	1
$\frac{3}{2} \gamma'^4$	$c^2 \gamma^2$	$2 \gamma (1 - 2 \gamma^2)$	2
	$B_{i'q'}$		i'
	$-\frac{1}{2} e'$		1
	$1 - \frac{5}{2} e'^2$		2
	$\frac{7}{2} e'$		3
	$\frac{17}{2} e'^2$		4
	$C_{iq} \quad (1 + \beta^2)^{-3}$	$\frac{dC_{iq}}{d\beta} \quad (1 + \beta^2)^{-4}$	i
	$- \beta^3$	$- 3 \beta^2 (1 - \beta^2)$	- 3
	$3 \beta^2 (1 + \beta^2)$	$6 \beta (1 - \beta^4)$	- 2
	$- 3 \beta (1 + 3 \beta^2 + \beta^4)$	$- 3 (1 - \beta^2) (1 + 5 \beta^2 + \beta^4)$	- 1
	$1 + 9 \beta^2 + 9 \beta^4 + \beta^6$	$12 \beta (1 - \beta^4)$	0
	$- 3 \beta (1 + 3 \beta^2 + \beta^4)$	$- 3 (1 - \beta^2) (1 + 5 \beta^2 + \beta^4)$	1
	$3 \beta^2 (1 + \beta^2)$	$6 \beta (1 - \beta^4)$	2
	$- \beta^3$	$- 3 \beta^2 (1 - \beta^2)$	3

ARBITRARY ECCENTRICITY

TABLE 5

$q = 2, \; q' = 2$

$A'_{\nu q'}$	$A_{\nu q}$	$\dfrac{dA_{\nu q}}{d\gamma}$	ν
$\frac{3}{4} c'^4$	γ^4	$4\gamma^3$	-2
$-3 c'^3 \gamma'$	$c\,\gamma^3$	$c^{-1}\gamma^2 (3 - 4\gamma^2)$	-1
$\frac{9}{2} c'^2 \gamma'^2$	$c^2 \gamma^2$	$2\gamma (1 - 2\gamma^2)$	0
$-3 c' \gamma'^3$	$c^3 \gamma$	$c\,(1 - 4\gamma^2)$	1
$\frac{3}{4}\gamma'^4$	c^4	$-4 c^2 \gamma$	2
	$B_{i'q'}$		i'
	$-\frac{1}{2} e'$		1
	$1 - \frac{5}{2} e'^2$		2
	$\frac{7}{2} e'$		3
	$\frac{17}{2} e'^2$		4
	$C_{iq} \quad (1 + \beta^2)^{-3}$	$\dfrac{dC_{iq}}{d\beta} \quad (1 + \beta^2)^{-4}$	i
	$-\beta^5$	$-5\beta^4 + \beta^2$	-3
	$\beta^4 (5 + \beta^2)$	$4\beta^3 (5 - \beta^2)$	-2
	$-\beta^3 (10 + 5\beta^2)$	$5\beta^2 (-2 + \beta^2)(3 + \beta^2)$	-1
	$10\beta^2 (1 + \beta^2)$	$20\beta (1 - \beta^4)$	0
	$-5\beta (1 + 2\beta^2)$	$-5 (1 - 2\beta^2)(1 + 3\beta^2)$	1
	$1 + 5\beta^2$	$4\beta (1 - 5\beta^2)$	2
	$-\beta$	$-1 + 5\beta^2$	3

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TABLE 6
$q = 1,\ q' = 1$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{45}{8} c'^4 \gamma'^2$	$c^2 \gamma^4$	$2 \gamma^3 (2 - 3 \gamma^2)$	-3
$\frac{15}{4} c'^3 (1 - 3 \gamma'^2) \gamma'$	$c \gamma^3 (2 - 3 \gamma^2)$	$c^{-1} \gamma^2 (6 - 23 \gamma^2 + 18 \gamma^4)$	-2
$\frac{3}{8} c'^2 (1 - 10 \gamma'^2 + 15 \gamma'^4)$	$\gamma^2 (6 - 20 \gamma^2 + 15 \gamma^4)$	$2 \gamma (6 - 40 \gamma^2 + 45 \gamma^4)$	-1
$-\frac{9}{2} c' \gamma' (1 - 5 \gamma'^2 + 5 \gamma'^4)$	$c \gamma (1 - 5 \gamma^2 + 5 \gamma^4)$	$c^{-1} (1 - 17 \gamma^2 + 45 \gamma^4 - 30 \gamma^6)$	0
$\frac{3}{8} \gamma'^2 (6 - 20 \gamma'^2 + 15 \gamma'^4)$	$c^2 (1 - 10 \gamma^2 + 15 \gamma^4)$	$-2 \gamma (11 - 50 \gamma^2 + 45 \gamma^4)$	1
$\frac{15}{4} c' \gamma'^3 (2 - 3 \gamma'^2)$	$c^3 (1 - 3 \gamma^2) \gamma$	$c (1 - 13 \gamma^2 + 18 \gamma^4)$	2
$\frac{45}{8} c'^2 \gamma'^4$	$c^4 \gamma^2$	$2 c^2 \gamma (1 - 3 \gamma^2)$	3
	$B_{i'q'}$		i'
	$\frac{11}{8} e'^2$		-1
	e'		0^*
	$1 + 2 e'^2$		1
	$3 e'$		2
	$\frac{53}{8} e'^2$		3
	C_{iq} $(1 + \beta^2)^{-4}$	$\frac{dC_{iq}}{d\beta}$ $(1 + \beta^2)^{-5}$	i
	$-\beta^5$	$-\beta^4 (5 - 3 \beta^2)$	-4
	$\beta^4 (5 + 3 \beta^2)$	$2 \beta^3 (10 - \beta^2 - 3 \beta^4)$	-3
	$-\beta^3 (10 + 15 \beta^2 + 3 \beta^4)$	$-\beta^2 (30 + 25 \beta^2 - 24 \beta^4 - 3 \beta^6)$	-2
	$\beta^2 (10 + 30 \beta^2 + 15 \beta^4 + \beta^6)$	$2 \beta (10 + 30 \beta^2 - 15 \beta^4 - 11 \beta^6)$	-1
	$-5 \beta (1 + 6 \beta^2 + 6 \beta^4 + \beta^6)$	$-5 (1 - \beta^4)(1 + 11 \beta^2 + \beta^4)$	0^*
	$1 + 15 \beta^2 + 30 \beta^4 + 10 \beta^6$	$2 \beta (11 + 15 \beta^2 - 30 \beta^4 - 10 \beta^6)$	1
	$-\beta (3 + 15 \beta^2 + 10 \beta^4)$	$-3 - 2 \beta^2 + 25 \beta^4 + 30 \beta^6$	2
	$\beta^2 (3 + 5 \beta^2)$	$2 \beta (3 + \beta^2 - 10 \beta^4)$	3
	$-\beta^3$	$-\beta^2 (3 - 5 \beta^2)$	4

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TABLE 7

$q = 1,\ q' = -1$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{45}{8} c'^2 \gamma'^4$	$c^2 \gamma^4$	$2\gamma^3 (2 - 3\gamma^2)$	-3
$\frac{15}{4} c' \gamma'^3 (2 - 3\gamma'^2)$	$c \gamma^3 (2 - 3\gamma^2)$	$c^{-1} \gamma^2 (6 - 23\gamma^2 + 18\gamma^4)$	-2
$\frac{3}{8} \gamma'^2 (6 - 20\gamma'^2 + 15\gamma'^4)$	$\gamma^2 (6 - 20\gamma^2 + 15\gamma^4)$	$2\gamma (6 - 40\gamma^2 + 45\gamma^4)$	-1
$\frac{9}{2} c' \gamma' (1 - 5\gamma'^2 + 5\gamma'^4)$	$c \gamma (1 - 5\gamma^2 + 5\gamma^4)$	$c^{-1} (1 - 17\gamma^2 + 45\gamma^4 - 30\gamma^6)$	0
$\frac{3}{8} c'^2 (1 - 10\gamma'^2 + 15\gamma'^4)$	$c^2 (1 - 10\gamma^2 + 15\gamma^4)$	$-2\gamma (11 - 50\gamma^2 + 45\gamma^4)$	1
$\frac{15}{4} c'^3 (1 - 3\gamma'^2) \gamma'$	$c^3 (1 - 3\gamma^2) \gamma$	$c (1 - 13\gamma^2 + 18\gamma^4)$	2
$\frac{45}{8} c'^4 \gamma'^2$	$c^4 \gamma^2$	$2c^2 \gamma (1 - 3\gamma^2)$	3
	$B_{i'q'}$		i'
	$\frac{53}{8} e'^2$		-3
	$3e$		-2
	$1 + 2e'^2$		-1
	e'		0^*
	$\frac{11}{8} e'^2$		1
	C_{iq} $(1 + \beta^2)^{-4}$	$\frac{dC_{iq}}{d\beta}$ $(1 + \beta^2)^{-5}$	i
	$-\beta^5$	$-\beta^4 (5 - 3\beta^2)$	-4
	$\beta^4 (5 + 3\beta^2)$	$2\beta^3 (10 - \beta^2 - 3\beta^4)$	-3
	$-\beta^3 (10 + 15\beta^2 + 3\beta^4)$	$-\beta^2 (30 + 25\beta^2 - 24\beta^4 - 3\beta^6)$	-2
	$\beta^2 (10 + 30\beta^2 + 15\beta^4 + \beta^6)$	$2\beta (10 + 30\beta^2 - 15\beta^4 - 11\beta^6)$	-1
	$-5\beta (1 + 6\beta^2 + 6\beta^4 + \beta^6)$	$-5 (1 - \beta^4)(1 + 11\beta^2 + \beta^4)$	0^*
	$1 + 15\beta^2 + 30\beta^4 + 10\beta^6$	$2\beta (11 + 15\beta^2 - 30\beta^4 - 10\beta^6)$	1
	$-\beta (3 + 15\beta^2 + 10\beta^4)$	$-3 - 24\beta^2 + 25\beta^4 + 30\beta^6$	2
	$\beta^2 (3 + 5\beta^2)$	$2\beta (3 + \beta^2 - 10\beta^4)$	3
	$-\beta^3$	$-\beta^2 (3 - 5\beta^2)$	4

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TABLE 8

$q = 1, \; q' = 3$

$A'_{\nu q'}$	$A_{\nu q}$	$\dfrac{dA_{\nu q}}{d\gamma}$	ν
$\frac{15}{8} c'^6$	$c^2 \gamma^4$	$2 \gamma^3 (2 - 3 \gamma^2)$	-3
$-\frac{15}{4} c'^5 \gamma'$	$c \gamma^3 (2 - 3 \gamma^2)$	$c^{-1} \gamma^2 (6 - 23 \gamma^2 + 18 \gamma^4)$	-2
$\frac{15}{8} c'^4 \gamma'^2$	$\gamma^2 (6 - 20 \gamma^2 + 15 \gamma^4)$	$2 \gamma (6 - 40 \gamma^2 + 45 \gamma^4)$	-1
$-\frac{15}{2} c'^3 \gamma'^3$	$c \gamma (1 - 5 \gamma^2 + 5 \gamma^4)$	$c^{-1} (1 - 17 \gamma^2 + 45 \gamma^4 - 30 \gamma^6)$	0
$\frac{15}{8} c'^2 \gamma'^4$	$c^2 (1 - 10 \gamma^2 + 15 \gamma^4)$	$-2 \gamma (11 - 50 \gamma^2 + 45 \gamma^4)$	1
$\frac{15}{4} c' \gamma'^5$	$c^3 \gamma (1 - 3 \gamma^2)$	$c (1 - 13 \gamma^2 + 18 \gamma^4)$	2
$\frac{15}{8} \gamma'^6$	$c^4 \gamma^2$	$2 c^2 \gamma$	3
	$B_{i'q'}$		i'
	$\frac{1}{8} e'^2$		1
	$-e'$		2
	$1 - 6 e'^2$		3
	$5 e'$		4
	$\frac{127}{8} e'^2$		5
	$C_{iq} \quad (1 + \beta^2)^{-4}$	$\dfrac{dC_{iq}}{d\beta} \quad (1 + \beta^2)^{-5}$	i
	$-\beta^5$	$-\beta^4 (5 - 3 \beta^2)$	-4
	$\beta^4 (5 + 3 \beta^2)$	$2 \beta^3 (10 - \beta^2 - 3 \beta^4)$	-3
	$-\beta^3 (10 + 15 \beta^2 + 3 \beta^4)$	$-\beta^2 (30 + 25 \beta^2 - 24 \beta^4 - 3 \beta^6)$	-2
	$\beta^2 (10 + 30 \beta^2 + 15 \beta^4 + \beta^6)$	$2 \beta (10 + 30 \beta^2 - 15 \beta^4 - 11 \beta^6)$	-1
	$-5 \beta (1 + 6 \beta^2 + 6 \beta^4 + \beta^6)$	$-5 (1 - \beta^4) (1 + 11 \beta^2 + \beta^4)$	0
	$1 + 15 \beta^2 + 30 \beta^4 + 10 \beta^6$	$2 \beta (11 + 15 \beta^2 - 30 \beta^4 - 10 \beta^6)$	1
	$-\beta (3 + 15 \beta^2 + 10 \beta^4)$	$-3 - 24 \beta^2 + 25 \beta^4 + 30 \beta^6$	2
	$\beta^2 (3 + 5 \beta^2)$	$2 \beta (3 + \beta^2 - 10 \beta^4)$	3
	$-\beta^3$	$-\beta^2 (3 - 5 \beta^2)$	4

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TABLE 9

$q = 1, \; q' = -3$

$A'_{\nu\, q'}$	$A_{\nu q}$	$\dfrac{d A_{\nu q}}{d\gamma}$	ν
$+\frac{15}{8}\gamma'^6$	$c^2\,\gamma^4$	$2\,\gamma^3\,(2 - 3\,\gamma^2)$	-3
$\frac{15}{4}c'\,\gamma'^5$	$c\,\gamma^3\,(2 - 3\,\gamma^2)$	$c^{-1}\,\gamma^2\,(6 - 23\,\gamma^2 + 18\,\gamma^4)$	-2
$\frac{15}{8}c'^2\,\gamma'^4$	$\gamma^2\,(6 - 20\,\gamma^2 + 15\,\gamma^4)$	$2\,\gamma\,(6 - 40\,\gamma^2 + 45\,\gamma^4)$	-1
$\frac{15}{2}c'^3\,\gamma'^3$	$c\,\gamma\,(1 - 5\,\gamma^2 + 5\,\gamma^4)$	$c^{-1}\,(1 - 17\,\gamma^2 + 45\,\gamma^4 - 30\,\gamma^6)$	0
$\frac{15}{8}c'^4\,\gamma'^2$	$c^2\,(1 - 10\,\gamma^2 + 15\,\gamma^4)$	$-2\,\gamma\,(11 - 50\,\gamma^2 + 45\,\gamma^4)$	1
$-\frac{15}{4}c'^5\,\gamma'$	$c^3\,\gamma\,(1 - 3\,\gamma^2)$	$c\,(1 - 13\,\gamma^2 + 18\,\gamma^4)$	2
$\frac{15}{8}c'^6$	$c^4\,\gamma^2$	$2\,c^2\,\gamma\,(1 - 3\,\gamma^2)$	3
	$B_{i'q'}$		i'
	$\frac{127}{8}e'^2$		-5
	$5\,e'$		-4
	$1 - 6\,e'^2$		-3
	$-e'$		-2
	$\frac{1}{8}e'^2$		-1
	C_{iq} $\quad (1 + \beta^2)^{-4}$	$\dfrac{dC_{iq}}{d\beta}$ $\quad (1 + \beta^2)^{-5}$	i
	$-\beta^5$	$-\beta^4\,(5 - 3\,\beta^2)$	-4
	$\beta^4\,(5 + 3\,\beta^2)$	$2\,\beta^3\,(10 - \beta^2 - 3\,\beta^4)$	-3
	$-\beta^3\,(10 + 15\,\beta^2 + 3\,\beta^4)$	$-\beta^2\,(30 + 25\,\beta^2 - 24\,\beta^4 - 3\,\beta^6)$	-2
	$\beta^2\,(10 + 30\,\beta^2 + 15\,\beta^4 + \beta^6)$	$2\,\beta\,(10 + 30\,\beta^2 - 15\,\beta^4 - 11\,\beta^6)$	-1
	$-5\,\beta\,(1 + 6\,\beta^2 + 6\,\beta^4 + \beta^6)$	$-5\,(1 - \beta^4)\,(1 + 11\,\beta^2 + \beta^4)$	0
	$1 + 15\,\beta^2 + 30\,\beta^4 + 10\,\beta^6$	$2\,\beta\,(11 + 15\,\beta^2 - 30\,\beta^4 - 10\,\beta^6)$	1
	$-\beta\,(3 + 15\,\beta^2 + 10\,\beta^4)$	$-3 - 24\,\beta^2 + 25\,\beta^4 + 30\,\beta^6$	2
	$\beta^2\,(3 + 5\,\beta^2)$	$2\,\beta\,(3 + \beta^2 - 10\,\beta^4)$	3
	$-\beta^3$	$-\beta^2\,(3 - 5\,\beta^2)$	4

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TABLE 10

$q = 3,\ q' = 1$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{15}{8} c'^4\, \gamma'^2$	γ^6	$6\, \gamma^5$	-3
$\frac{15}{4} c'^3\, \gamma' \,(1 - 3\, \gamma'^2)$	$c\, \gamma^5$	$c^{-1}\, \gamma^4\, (5 - 6\, \gamma^2)$	-2
$\frac{15}{8} c'^2\, (1 - 10\, \gamma'^2 + 15\, \gamma'^4)$	$c^2\, \gamma^4$	$2\, \gamma^3\, (2 - 3\, \gamma^2)$	-1
$-\frac{15}{2} c'\, \gamma'\, (1 - 5\, \gamma'^2 + 5\, \gamma'^4)$	$c^3\, \gamma^3$	$3\, c\, \gamma^2\, (1 - 2\, \gamma^2)$	0
$\frac{15}{8} \gamma'^2\, (6 - 20\, \gamma'^2 + 15\, \gamma'^4)$	$c^4\, \gamma^2$	$2\, c^2\, \gamma\, (1 - 3\, \gamma^2)$	1
$-\frac{15}{4} c'\, \gamma'^3\, (2 - 3\, \gamma'^2)$	$c^5\, \gamma$	$c^3\, (1 - 6\, \gamma^2)$	2
$\frac{15}{8} c'^2\, \gamma'^4$	c^6	$-6\, c^4\, \gamma$	3
	$B_{i'q'}$		i'
	$\frac{11}{8} e'^2$		-1
	e'		0*
	$1 + 2\, e'^2$		1
	$3\, e'$		2
	$\frac{53}{8} e'^2$		3
	$C_{iq} \quad (1 + \beta^2)^{-4}$	$\frac{dC_{iq}}{d\beta} \quad (1 + \beta^2)^{-5}$	i
	$-\beta^7$	$-\beta^6\, (7 - \beta^2)$	-4
	$\beta^6\, (7 + \beta^2)$	$6\, \beta^5\, (7 - \beta^2)$	-3
	$-7\, \beta^5\, (3 + \beta^2)$	$-7\, \beta^4\, (5 + \beta^2)\, (3 - \beta^2)$	-2
	$7\, \beta^4\, (5 + 3\, \beta^2)$	$14\, \beta^3\, (10 - \beta^2 - 3\, \beta^4)$	-1
	$-35\, \beta^3\, (1 + \beta^2)$	$-105\, \beta^2\, (1 - \beta^4)$	0*
	$7\, \beta^2\, (3 + 5\, \beta^2)$	$14\, \beta\, (3 + \beta^2 - 10\, \beta^4)$	1
	$-7\, \beta\, (1 + 3\, \beta^2)$	$-7\, (1 - 3\, \beta^2)\, (1 + 5\, \beta^2)$	2
	$1 + 7\, \beta^2$	$6\, \beta\, (1 - 7\, \beta^2)$	3
	$-\beta$	$-(1 - 7\, \beta^2)$	4

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TABLE 11

$q = 3,\ q' = -1$

$A'_{\nu q'}$	$A_{\nu q}$	$\frac{dA_{\nu q}}{d\gamma}$	ν
$\frac{15}{8} e'^2\ \gamma'^4$	γ^6	$6\ \gamma^5$	-3
$\frac{15}{4} e'\ \gamma'^3\ (2 - 3\ \gamma'^2)$	$e\ \gamma^5$	$e^{-1}\ \gamma^4\ (5 - 6\ \gamma^2)$	-2
$\frac{15}{8} \gamma'^2\ (6 - 20\ \gamma'^2 + 15\ \gamma'^4)$	$e^2\ \gamma^4$	$2\ \gamma^3\ (2 - 3\ \gamma^2)$	-1
$\frac{15}{2} e'\ \gamma'\ (1 - 5\ \gamma'^2 + 5\ \gamma'^4)$	$e^3\ \gamma^3$	$3\ e\ \gamma^2\ (1 - 2\ \gamma^2)$	0
$\frac{15}{8} e'^2\ (1 - 10\ \gamma'^2 + 15\ \gamma'^4)$	$e^4\ \gamma^2$	$2\ e^2\ \gamma\ (1 - 3\ \gamma^2)$	1
$-\frac{15}{4} e'^3\ \gamma'\ (1 - 3\ \gamma'^2)$	$e^5\ \gamma$	$e^3\ (1 - 6\ \gamma^2)$	2
$\frac{15}{8} e'^4\ \gamma'^2$	e^6	$-6\ e^4\ \gamma$	3
	$B_{i' q'}$		i'
	$\frac{53}{8} e'^2$		-3
	$3\ e'$		-2
	$1 + 2\ e'^2$		-1
	e'		0*
	$\frac{11}{8} e'^2$		1
	$C_{iq}\quad (1 + \beta^2)^-$	$\frac{dC_{iq}}{d\beta}\qquad (1 + \beta^2)^{-5}$	i
	$-\beta^7$	$-\beta^6\ (7 - \beta^2)$	-4
	$\beta^6\ (7 + \beta^2)$	$6\ \beta^5\ (7 - \beta^2)$	-3
	$-7\ \beta^5\ (3 + \beta^2)$	$-7\ \beta^4\ (5 + \beta^2)\ (3 - \beta^2)$	-2
	$7\ \beta^4\ (5 + 3\ \beta^2)$	$14\ \beta^3\ (10 - \beta^2 - 3\ \beta^4)$	-1
	$-35\ \beta^3\ (1 + \beta^2)$	$-105\ \beta^2\ (1 - \beta^4)$	0*
	$7\ \beta^2\ (3 + 5\ \beta^2)$	$14\ \beta\ (3 + \beta^2 - 10\ \beta^4)$	1
	$-7\ \beta\ (1 + 3\ \beta^2)$	$-7\ (1 - 3\ \beta^2)\ (1 + 5\ \beta^2)$	2
	$1 + 7\ \beta^2$	$6\ \beta\ (1 - 7\ \beta^2)$	3
	$-\beta$	$-(1 - 7\ \beta^2)$	4

TABLE 12
$q = 3, \quad q' = 3$

$A'_{\nu q'}$	$A_{\nu q}$	$\dfrac{dA_{\nu q}}{d\gamma}$	ν
$\frac{5}{8} c'^6$	γ^6	$6\,\gamma^5$	-3
$-\frac{15}{4} c'^5\,\gamma'$	$c\,\gamma^5$	$c^{-1}\,\gamma^4\,(5 - 6\,\gamma^2)$	-2
$\frac{75}{8} c'^4\,\gamma'^2$	$c^2\,\gamma^4$	$2\,\gamma^3\,(2 - 3\,\gamma^2)$	-1
$-\frac{25}{2} c'^3\,\gamma'^3$	$c^3\,\gamma^3$	$3\,c\,\gamma^2\,(1 - 2\,\gamma^2)$	0
$\frac{75}{8} c'^2\,\gamma'^4$	$c^4\,\gamma^2$	$2\,c^2\,\gamma\,(1 - 3\,\gamma^2)$	1
$-\frac{15}{4} c'\,\gamma'^5$	$c^5\,\gamma$	$c^3\,(1 - 6\,\gamma^2)$	2
$\frac{5}{8} \gamma'^6$	c^6	$-6\,c^4\,\gamma$	3
	$B_{i'q'}$		i'
	$\frac{1}{8} e'^2$		1
	$-e'$		2
	$1 - 6\,e'^2$		3
	$5\,e'$		4
	$\frac{127}{8} e'^2$		5
	$C_{iq} \quad (1 + \beta^2)^{-4}$	$\dfrac{dC_{iq}}{d\beta} \quad (1 + \beta^2)^{-5}$	i
	$-\beta^7$	$-\beta^6\,(7 - \beta^2)$	-4
	$\beta^6\,(7 + \beta^2)$	$6\,\beta^5\,(7 - \beta^2)$	-3
	$-7\,\beta^5\,(3 + \beta^2)$	$-7\,\beta^4\,(5 + \beta^2)\,(3 - \beta^2)$	-2
	$7\,\beta^4\,(5 + 3\,\beta^2)$	$14\,\beta^3\,(10 - \beta^2 - 3\,\beta^4)$	-1
	$-35\,\beta^3\,(1 + \beta^2)$	$-105\,\beta^2\,(1 - \beta^4)$	0
	$7\,\beta^2\,(3 + 5\,\beta^2)$	$14\,\beta\,(3 + \beta^2 - 10\,\beta^4)$	1
	$-7\,\beta\,(1 + 3\,\beta^2)$	$-7\,(1 - 3\,\beta^2)\,(1 + 5\,\beta^2)$	2
	$1 + 7\,\beta^2$	$6\,\beta\,(1 - 7\,\beta^2)$	3
	$-\beta$	$-(1 - 7\,\beta^2)$	4

TABLE 13

$q = 3,\ q' = -3$

$A'_{\nu q'}$	$A_{\nu q}$	$\dfrac{dA_{\nu q}}{d\gamma}$	ν
$\frac{5}{8}\gamma'^6$	γ^6	$6\gamma^5$	-3
$\frac{15}{4}e'\gamma'^5$	$e\gamma^5$	$e^{-1}\gamma^4(5-6\gamma^2)$	-2
$\frac{75}{8}e'^2\gamma'^4$	$e^2\gamma^4$	$2\gamma^3(2-3\gamma^2)$	-1
$\frac{25}{2}e'^3\gamma'^3$	$e^3\gamma^3$	$3e\gamma^2(1-2\gamma^2)$	0
$\frac{75}{8}e'^4\gamma'^2$	$e^4\gamma^2$	$2e^2\gamma(1-3\gamma^2)$	1
$\frac{15}{4}e'^5\gamma'$	$e^5\gamma$	$e^3(1-6\gamma^2)$	2
$\frac{5}{8}e'^6$	e^6	$-6e^4\gamma$	3
	$B_{i'q'}$		i'
	$\frac{127}{8}e'^2$		-5
	$5e'$		-4
	$1-6e'^2$		-3
	$-e'$		-2
	$\frac{1}{8}e'^2$		-1
	$C_{iq}\quad(1+\beta^2)^{-4}$	$\dfrac{dC_{iq}}{d\beta}\quad(1+\beta^2)^{-5}$	i
	$-\beta^7$	$-\beta^6(7-\beta^2)$	-4
	$\beta^6(7+\beta^2)$	$6\beta^5(7-\beta^2)$	-3
	$-7\beta^5(3+\beta^2)$	$-7\beta^4(5+\beta^2)(3-\beta^2)$	-2
	$7\beta^4(5+3\beta^2)$	$14\beta^3(10-\beta^2-3\beta^4)$	-1
	$-35\beta^3(1+\beta^2)$	$-105\beta^2(1-\beta^4)$	0
	$7\beta^2(3+5\beta^2)$	$14\beta(3+\beta^2-10\beta^4)$	1
	$-7\beta(1+3\beta^2)$	$-7(1-3\beta^2)(1+5\beta^2)$	2
	$1+7\beta^2$	$6\beta(1-7\beta^2)$	3
	$-\beta$	$-(1-7\beta^2)$	4

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TABLE 14

$$Q = \frac{A}{ni + n'i'} \left(Q_0' + me\, Q_1' + m^2 e^2 Q_2' \right)$$

$$Q_0' = \sin\left(i\,E + i'\,\ell' + \phi \right)$$

$$Q_1' = \frac{i' \sin\left[(i+1)\,E + i'\,\ell' + \phi \right]}{2\left[(i+1) + i'\,m \right]} + \frac{i' \sin\left[(i-1)\,E + i'\,\ell' + \phi \right]}{2\left[(i-1) + i'\,m \right]}$$

$$Q_2' = \frac{i'^2 \sin\left[(i-2)\,E + i'\,\ell' + \phi \right]}{4\left[(i-1) + i'\,m \right]\left[(i+1) + i'\,m \right]} + \frac{i'^2 \sin\left(i\,E + i'\ell' + \phi \right)}{2\left[(i-1) + i'm \right]\left[(i+1) + i'm \right]}$$

$$+ \frac{i'^2 \sin\left[(i+2)\,E + i'\ell' + \phi \right]}{4\left[(i+1) + i'm \right]\left[(i+2) + i'm \right]}$$

$$\frac{\partial Q_0'}{\partial E} = i \cos\left(i\,E + i'\,\ell' + \phi \right)$$

$$\frac{\partial Q_0'}{\partial \omega} = q \cos\left(i\,E + i'\,\ell' + \phi \right)$$

$$\frac{\partial Q_0'}{\partial \theta} = \nu \cos\left(i\,E + i'\,\ell' + \phi \right)$$

$$\frac{\partial Q_1'}{\partial E} = \frac{i'\,(i+1) \cos\left[(i+1)\,E + i'\,\ell' + \phi \right]}{2\left[(i+1) + i'\,m \right]} + \frac{i'\,(i-1) \cos\left[(i-1)\,E + i'\,\ell' + \phi \right]}{2\left[(i-1) + i'\,m \right]}$$

$$\frac{\partial Q_1'}{\partial \omega} = \frac{i'\,q \cos\left[(i+1)\,E + i'\,\ell' + \phi \right]}{2\left[(i+1) + i'\,m \right]} \quad \frac{i'\,q \cos\left[(i-1)\,E + i'\,\ell' + \phi \right]}{2\left[(i-1) + i'\,m \right]}$$

$$\frac{\partial Q_1'}{\partial \theta} = \frac{i'\,\nu \cos\left[(i+1) + i'\,\ell' + \phi \right]}{2\left[(i+1) + i'\,m \right]} \quad \frac{i\,\nu \cos\left[(i-1)\,E + i'\,\ell' + \phi \right]}{2\left[(i-1) + i'\,m \right]}$$

TABLE 14 (Continued)

$$\frac{\partial Q_2'}{\partial E} = \frac{i'^2\,(i-2)\cos\,[(i-2)\,E + i'\,\ell' + \phi]}{4\,[(i-1)+i'm]\,[(i-2)+i'm]} + \frac{i'^3\cos\,(i\,E + i'\,\ell' + \phi)}{2\,[(i-1)+i'm]\,[(i+1)+i'm]}$$

$$+ \frac{i'^2\,(i+2)\cos\,[(i+2)\,E + i'\,\ell' + \phi]}{4\,[(i+1)+i'm]\,[(i+2)+i'm]}$$

$$\frac{\partial Q_2'}{\partial \omega} = \frac{q\,i'^2\cos\,[(i-2)\,E + i'\,\ell' + \phi]}{4\,[(i-1)+i'm]\,[(i-2)+i'm]} + \frac{q\,i'^2\cos\,(i\,E + i'\,\ell' + \phi)}{2\,[(i-1)+i'm]\,[(i+1)+i'm]}$$

$$+ \frac{q\,i'^2\cos\,[(i+2)\,E + i'\,\ell' + \phi]}{4\,[(i+1)+i'm]\,[(i+2)+i'm]}$$

$$\frac{\partial Q_2'}{\partial \theta} = \frac{\nu\,i'^2\cos\,[(i-2)\,E + i'\,\ell' + \phi]}{4\,[(i-1)+i'm]\,[(i-2)+i'm]} + \frac{\nu\,i'^2\cos\,(i\,E + i'\,\ell' + \phi)}{2\,[(i-1)+i'm]\,[(i+1)+i'm]}$$

$$+ \frac{\nu\,i'^2\cos\,[(i+2)\,E + i'\,\ell' + \phi]}{4\,[(i+1)+i'm]\,[(i+2)+i'm]}$$

$$\frac{\partial\,[meQ_1']}{\partial a} = \frac{3\,i'\,(i+1)\,me\sin\,[(i+1)\,E + i'\,\ell' + \phi]}{4\,a\,[(i+1)+i'm]^2}$$

$$+ \frac{3\,i'\,(i-1)\,me\sin\,[(i-1)\,E + i'\,\ell' + \phi]}{4\,a\,[(i-1)+i'm]^2}$$

TABLE 14 (Continued)

$$\frac{\partial\,[m^2 e^2 Q_2']}{\partial a} = \frac{3\,i'^2 m^2 e^2\,[2(i-1)(i-2) + i'(2i-3)m]\sin[(i-2)E + i'\ell' + \phi]}{8a\,[(i-1)+i'm]^2\,[(i-2)+i'm]^2}$$

$$+\frac{3\,i'^2 m^2 e^2\,[i^2 - 1 + m\,i\,i']\sin(i\,E + i'\ell' + \phi)}{2a\,[(i-1)+i'm]^2\,[(i+1)+i'm]^2}$$

$$+\frac{3\,i'^2 m^2 e^2\,[2(i+1)(i+2) + i'(2i+3)m]\sin[(i+2)E + i'\ell' + \phi]}{8a\,[(i+1)+i'm]^2\,[(i-2)+i'm]^2}$$

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